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Generalization of numerical range of polynomial operator matrices

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ABSTRACT

Suppose that $Q(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0$ is a polynomial matrix operator where $A_i \in M_n(\mathbb{C})$ for $i = 0,1,\ldots,m$, are $n \times n$ complex matrix and let λ be a complex variable. For an $n \times n$ Hermitian matrix S, we define the V-numerical range of polynomial matrix of $Q(\lambda)$ as $V_S(Q(\lambda)) = \{\lambda \in \mathbb{C};\ \langle Q(\lambda)x, x\rangle = 0, \text{ for some } x \in \mathbb{C}^n,\ \langle x, x\rangle_S \neq 0\}$, where $\langle x, y\rangle_S = y^* S x$. In this paper we study $V_S(Q(\lambda))$ and our emphasis is on the geometrical properties of $V_S(Q(\lambda))$. We consider the location of $V_S(Q(\lambda))$ in the complex plane and a theorem concerning the boundary of $V_S(Q(\lambda))$ is also obtained. Possible generalazations of our results including their extensions to bounded linerar operators on an infinite dimensional Hilbert space are described.

1. Introduction

Let $A \in M_n(\mathbb{C})$. The numerical range (the field of value) of A defined and denoted by $W(A) = \{x^* A x : x \in \mathbb{C}^n, x^* x = 1\}$
where x^* is the conjugate transpose of x. Many researchers have dedicated their work to the study of numerical ranges not only in Hilbert spaces but also in Banach spaces and Banach Algebras, because it is very useful tool in studying and understanding the spectral analysis of unbounded and bounded linear operators in Hilbert spaces as explained in most functional analysis and matrix analysis textbooks [15],[16] and [17].

We can see that the numerical range is the image of the Euclidean unit ball in $\mathbb{C}^n$, which is a compact and connected set, under the continuous mapping $x \to x^* A x$. Therefore, $W(A)$ is a compact and connected set. $W(A)$ is also a convex set. For infinite dimensional Hilbert space we have the following: Let H be a complex Hilbert space with inner product $\langle .,. \rangle$ and let $B(H)$ be the algebra of all bounded linear operators on H. For $A \in B(H)$, the numerical range of A is the set

$$W(A) = \{u \in \mathbb{C} : \langle Au, u\rangle, \text{ for some } u \in H, ||u|| = 1\}.$$

In this context, $W(A)$ is convex, bounded but not necessarily closed. For properties on numerical range on infinite dimensional Hilbert spaces we refer to [18] and [19].

The concept of numerical range (the field of value) has been generalized in several directions, some of the generalizations that relevant to this study. The notation of the numerical range of matrix polynomials was first introduce by P. H. Muller in 1954 [11]. It has been systematically studied over the last decade. Moreover several of interesting results have been obtained (see e.g., [12], [13] and [14]).

Suppose that $M_n(\mathbb{C})$ is the algebra of all $n \times n$ complex matrices, and consider the polynomial matrix

$$Q(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_0 \qquad (1)$$

with $A_i \in M_n(\mathbb{C})$ for $i = 0,1,\ldots,m$, and λ is a complex variable. Define the numerical range of $Q(\lambda)$ as

$$W(Q(\lambda)) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x, x\rangle = 0, \text{ for some nonzero } x \in \mathbb{C}^n, \langle x, x\rangle = 1\}, \qquad (2)$$

the concept coincides to the well-known (classical) numerical range when $Q(\lambda) = \lambda I - A$, defined by

$W(A) = \{x^*Ax, \text{for some } x \in \mathbb{C}^n, x^*x = 1\}$. (3)
In the study of matrices and operators, the numerical range is a useful instrument that has been well researched [1] and [2].
Similar to the classical numerical range of A, the numerical range of a matrix polynomial is always closed and contains its spectrum $\sigma(Q) = \{\lambda \in \mathbb{C} : det\, Q(\lambda) = 0\}$.(e.g., see [3], [10] and its references)
Replacing in (2) the Euclidean inner product with the indefinite inner product on $\mathbb{C}^n$, it is known [3] that there exists an invertible Hermitian matrix S such that $\langle x, y\rangle_S = \langle Sx, y\rangle$. Hence, we introduced V- numerical range of matrix polynomial of $Q(\lambda)$ as
$V_S(Q(\lambda)) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x, x\rangle = 0, \text{for some } x \in \mathbb{C}^n, \langle x, x\rangle_S \neq 0\}$ (4)
which coincides with the positive V-numerical range of $Q(\lambda)$:
$V_S^+(Q(\lambda)) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x, x\rangle = 0, \text{for some } x \in \mathbb{C}^n, \langle x, x\rangle_S = 1\}$. (5)
Since S is positive definite, $S = X^*X$ for some nonsingular X and it is simple to prove that $V_S(Q(\lambda)) = V_S^+(Q(\lambda)) = W(X\,Q(\lambda)\,X^{-1})$. In particular, $V_S^+(Q(\lambda)) = W(S^{\frac{-1}{2}}Q(\lambda)S^{\frac{-1}{2}})$, where $S^{\frac{-1}{2}}$ denotes the inverse of $S^{\frac{1}{2}}$ and $S^{1/2}$ denotes the (unique) positive definite matrix that satisfies $(S^{1/2})^2 = S$.
Many properties of $W(Q(\lambda))$, can be extended to $V_S(Q(\lambda))$ and $V_S^+(Q(\lambda))$ if S is positive definite. Moreover, the concept coincides to the well-known V-numerical range when $Q(\lambda) = \lambda I - A$, defined by $V_S(A) = \left\{\frac{\langle Ax,x\rangle_S}{\langle x,x\rangle_S}, \text{for some } x \in \mathbb{C}^n, \langle x, x\rangle_S \neq 0\right\}$.
Currently under investigation is the numerical range of an operator defined on an infinite inner product space (see [1] and references therein). The fundamental characteristics of the classical numerical range are compactness and convexity. But $V_S(A)$ may be neither closed nor bounded in contrast to the classical situation. However, $V_S(A)$ is the union of two convex sets, where $V_S(A) = V_S^+(A) \cup V_S^-(A)$, even if it could not be convex. where
$V_S^+(A) = \{\langle Ax, x\rangle, \text{for some } x \in \mathbb{C}^n, \langle x, x\rangle_S = 1\}$
and
$V_S^-(A) = \{\langle Ax, x\rangle, \text{for some } x \in \mathbb{C}^n, \langle x, x\rangle_S = -1\}$
Bayasgalan [4] proved that if A is positive definite, its closure contains the spectrum of A. It is difficult to produce an accurate computer plot of this set because the numerical ranges of the matrix polynomials in an indefinite inner product space like $V_S(A)$ are neither bounded nor closed. The description of $V_S(Q(\lambda))$ for $A_i \in M_n(\mathbb{C})$ and $n > 2$ is mor complicated, so it would be useful to have a code that generates graphical representations. On the other hand, $V_S(Q(\lambda))$ may not be convex. During the recent decades, the S-numerical range and the numerical range of a matrix polynomial has been extensively studied by many researchers (see, [5], [6], [7], [8], [9]and the references therein).

This article is a first study of V-numerical range of matrix polynomials of an indefinite inner product space. In section 2 we discuss some fundamental properties of $V_S(Q\lambda))$ based on the results of [1] and [7], and we consider the location of V-numerical range of matrix polynomials in the complex plane and when it is bounded. On the other hand, we prove that the number of connected components of the V-numerical range of $Q(\lambda)$ in (1) does not excced m, and we study the boundary of $V_S(Q(\lambda))$. Furthermore, in section 3 we extend the definition of $V_S(Q(\lambda))$ to V-numerical range polynomial operator of infinite dimensional Hilbert space.

2. Main Results

In the following proposition, we describe some fundamental properties of $V_S(Q(\lambda))$ that are easily verifiable.
Proposition 2.1: Let $Q(\lambda)$ be as in Eq. (1), $A_m \neq 0$ and S is a Hermitian matrix then
i) $V_S(Q(\lambda + \alpha)) = V_S(Q(\lambda)) - \alpha$ for any $\alpha \in \mathbb{C}$.
ii) $V_S(\alpha Q(\lambda)) = V_S(Q(\lambda))$ for any non-zero $\alpha \in \mathbb{C}$.
iii) If $x^*A_ix = 0$ for all i then $V_S(Q(\lambda))$ is a hole complex plane.
iv) For any unitary matrix U, $V_S(U^*Q(\lambda)U) = V_S(Q(\lambda))$ if and only if $U^*SU = S$.
v) If $P(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m$ then $V_S(P(\lambda)) \setminus \{0\} = \{\mu^{-1} \in \mathbb{C}: \mu \in V_S(Q(\lambda))\}$
The fundamental properties of the classical numerical range are compactness and convexity. Condition (iii) in Remark 2.2 demonstrates that $V_S(Q(\lambda))$ is closed if S is positive definite.
Remark 2.2: The same properties are holding when $V_S(Q(\lambda))$ is replaced by $V_S^{\pm}(Q(\lambda))$. Furthermore, for these sets we have additional properties.
i) If S is positive definite then $V_S^-(Q(\lambda))$ is empty.
ii) If S is negative definite then $V_S^+(Q(\lambda))$ is empty.
iii) $V_S^+(Q(\lambda))$ and $V_S^-(Q(\lambda))$ are closed sets.
The following example shows that, $V_S^+(Q(\lambda))$ need not be connected or bounded.
Example 2.3: Let $Q(\lambda) = \lambda^m \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then
$V_S^+(Q(\lambda)) = \{\lambda \in \mathbb{C}: \lambda^m q - 1 = 0 \text{ for some } q \in [-1,1]\}$
$= \{re^{i\theta}: r \geq 1 \text{ and } \theta = k\pi/m, k = 0,1, \dots, 2m - 1\}$
There are $(2m)$ unbounded connected components.
The following example shows that the convexity $V_S^+(Q(\lambda))$ is not necessary.
Example 2.4: Let $Q(\lambda) = \lambda^m \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then
$V_S^+(Q(\lambda)) = \{\lambda \in \mathbb{C}: \lambda^m - q = 0 \text{ for some } q \in [-1,1]\}$
$= \{re^{i\theta}: 0 \leq r \leq 1 \text{ and } \theta = k\pi/m, k = 0,1, \dots, 2m - 1\}$

has a single non-convex component.

Theorem 2.5: If S is a positive definite, then $V_S^+(Q(\lambda)) \subseteq W(Q(\lambda))$.

Proof: Let $\lambda \in V_S^+(Q(\lambda))$. Then there exsits S-unit vector $x \in \mathbb{C}^n$ such that $\langle Q(\lambda)x, x\rangle = 0$ and $\langle Sx, x\rangle = 1$. Since S is a positive definite then the positve definite matrix $S^{\frac{1}{2}}$ exist and $S^{\frac{1}{2}}S^{\frac{1}{2}} = S$. Choose $y = S^{\frac{1}{2}}x$ then it is clear $\langle y, y\rangle = 1$.

$0 = \langle Q(\lambda)x, x\rangle = \langle Q(\lambda)S^{\frac{-1}{2}}x, S^{\frac{-1}{2}}x\rangle =$

$\langle \left(S^{\frac{-1}{2}}\right)^* Q(\lambda)S^{\frac{-1}{2}}, x\rangle$

then by properties of $W(Q(\lambda))$ in $\lambda \in W(Q(\lambda))$.

Lemma 2.6: Suppose that $A \in M_n(\mathbb{C})$ and S is a $n \times n$ Hermitian matrix. Then $V_S^+(A_m)$ is either has interior point or singleton.

Proof: Suppose that $V_S^+(A_m)$ is neither has interior point nor singleton, then by convexity of $V_S^+(A_m)$ and Schur theorem we can assume that $V_S^+(A_m) = [\zeta, \eta] \subset \mathbb{R}$ and without loss of generality we consider the (2×2) case

$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$,

where (a, b, d) is not of the form $(k, 0, k)$ $(k \in \mathbb{C})$,and

$S = \begin{pmatrix} \alpha & \beta \\ \beta & 0 \end{pmatrix}$,

where α and β are non-zero real numbers. Then for any S-unit vectors $x = \begin{pmatrix} x \\ y \end{pmatrix}$ with $\alpha(\bar{x}x) + \beta(\bar{x}y) + \beta(\bar{y}x) = 1$ we have $x^*Ax = a(\bar{x}x) + b(\bar{x}y) + d(\bar{y}y) \quad = (\alpha - \frac{a\alpha}{\beta})|x|^2 + d|y|^2 - b(\bar{y}x) + \frac{b}{\beta}$.

Clearly we can find S-unit vector x such that $x^*Ax \in \mathbb{C} \setminus \mathbb{R}$. Thus, $V_S^+(A)$ cannot be a closed interval in $\mathbb{R}$.

The following Theorem shows that $V_S^+(Q(\lambda))$ is bounded if and only if zero does not belongs to $V_S^+(A_m)$.

Theorem 2.7: Let $Q(\lambda)$ be as in Eq. (1) and let S be a $n \times n$ Hermitian matrix. Then $V_S^+(Q(\lambda))$ is bounded if and only if $0 \notin V_S^+(A_m)$.

Proof: Consider $V_S^+(Q(\lambda))$ is bounded and $V_S^+(Q(\lambda)) = \{\lambda \in \mathbb{C}; x^*Q(\lambda)x = 0, \text{for some } x \in \mathbb{C}^n, x^*Sx = 1\}$, hence all roots of polynomial $x^*Q(\lambda)x = 0$ is bounded. Furthermore, the function $\frac{x^*A_kx}{x^*A_mx}$ is a member of every roots where $0 \le k < m$. So there exist $r \in \mathbb{R}$ such that $|\frac{x^*A_kx}{x^*A_mx}| < r$, a simple calculation we get that $|x^*A_mx| > q$ for $q \in \mathbb{R}$, by lemma 2.6 $0 \notin V_S^+(A_m)$. Conversely, suppose that $V_S^+(Q(\lambda))$ is unbounded then there exists $\mu \in V_S^+(Q(\lambda))$ such that $|\mu| > r$, for $r \in \mathbb{R}$. Moreover, the function $\frac{x^*A_lx}{x^*A_mx}$ is member of every roots where $0 \le l < m$. This implies that there exists $0 \le j < m$ such that the function $\frac{x^*A_jx}{x^*A_mx}$ is the member of all roots i.e. $|\frac{x^*A_jx}{x^*A_mx}| > r$, for $r \in \mathbb{R}$. A simple calculation we get $|x^*A_mx| < q$ for $q \in \mathbb{R}$, by lemma 2.6 $0 \in V_S^+(A_m)$.

The following result can be proved in a similar fashion as Theorem 2.7.

Corollary 2.8: Let $Q(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_0$ be a monic polynomial and let S be $n \times n$ a Hermitian matrix then $V_S^+(Q(\lambda))$ is bounded.

For monic polynomial $Q(\lambda)$, the positive V-numerical range of $Q(\lambda)$ is always bounded. Furthermore we define the inner V-numerical radius $\tilde{r}_s(A_0) = \min_{\langle Sx,x\rangle=1}|x^*A_0x|$ and the outer V-numerical radius $r_s(A_j) = \max_{\langle Sx,x\rangle=1}|x^*A_jx|$ then $V_S^+(Q(\lambda))$ is located in circular annulus.

Theorem 2.9: Suppose that $Q(\lambda)$ be a monic matrix polynomial and let $\lambda \in V_S^+(Q(\lambda))$ then

$r_1 \le |\lambda| \le 1 + r_2$

where

$r_1 = \frac{\tilde{r}_s(A_0)}{r_s(A_0)+\max_{k\neq 0} r_s(A_k)}$, $r_2 = \max\{r_s(A_k): k = 0,1, \dots, m-1\}$, $\tilde{r}_s(A_0)$ is inner V-numerical radius and $r_s(A_j)$ is outer V-numerical radius.

Proof: Let x be any S-unit vector of $\mathbb{C}^n$, the roots of polynomial

$x^*Q(\lambda)x = \lambda^m + x^*A_{m-1}x\lambda^{m-1} + \cdots + x^*A_0x \quad (6)$

lie in the disc $B_0(1 + R_k)$, where $R_k = \max\{r_s(A_k): k = 0,1, \dots, m-1\} \le \max_k r_s(A_k)$, then obviously

$V_S^+(Q(\lambda)) \subset B_0(1 + r_2)$

Furthermore, every roots of equation (6) lie on or outside the circle

$|\lambda| = \min_{k=1,2,\dots,m} \frac{|x^*A_0x|}{|x^*A_0x|+|x^*SA_kx|}$

and consequently we take

$|\lambda| \ge \frac{\min|x^*A_0x|}{\max|x^*A_0x|+\max_{k\neq 0}|x^*A_kx|}$

$= \frac{\tilde{r}_s(A_0)}{r_s(A_0)+\max_{k\neq 0} r_s(A_k)} = r_2$

The next theorem shows that the number of connected component of the positive V-numerical range of $Q(\lambda)$ does not exceed m.

Theorem 2.10: Let $Q(\lambda)$ be as in Eq. (1), where $A_m \neq 0$ and S be a $n \times n$ Hermitian matrix. Suppose $V_S^+(Q(\lambda))$ has r connected component.

1) Let s be the minimum number of distinct roots of the polynomial $x^*Q(\lambda)x$ with $x^*Sx = 1$ such that $x^*A_mx \neq 0$ and consider $V_S^+(A_m) \setminus \{0\}$ is connected. Then

$r \le s \le m$

2) If s_i is the minimum number of distinct roots of the polynomial $x^*Q(\lambda)x$ with $x^*Sx = 1$, and consider $V_S^+(A_m) \setminus \{0\}$ has disjoint connected component $\mathcal{G}_1$ and $\mathcal{G}_1$ such that $x^*A_mx \in \mathcal{G}_i$ for $i = 1,2$. Then

$r \le s_1 + s_2 \le 2m$

Proof: Consider $G = \{x \in \mathbb{C}^n, x^*Sx = 1\}$. Let $u, v \in V_S^+(A_m)$ then there exist S-unit vectors $x, y \in G$ such that $x^*A_mx = u$ and $y^*A_my = v$. Since $V_S^+(A_m)$ is convex set, so the line segment $[u, v]$ lies in $V_S^+(A_m)$.

i) If $0 \notin [u, v]$ then there exists a continuous curve $z: [0,1] \to G$ such that $z(0) = x$, $z(1) = \mu y$ with $|\mu| = 1$ and $z(t)^* A_m z(t) \in [u, v]$ for every $t \in [0,1]$

ii) If $0 \in [u, v]$ then we can choose $r \in V_S^+(A_m)$ such that $0 \notin [u, r] \cup [r, v]$. Again by using convexity of $V_S^+(A_m)$ there exists a continuous curve $z: [0,1] \to G$ such that $z(0) = x$, $z(1) = \mu y$ with $|\mu| = 1$ and $z(\frac{1}{2}) = r$, where $r = q^* A_m q$ and $q \in G$. So, $z(t)^* A_m z(t) \in [u, r] \cup [r, v]$ for every $t \in [0,1]$.

iii) Finally, suppose $u = v$. If $V_S^+(A_m)$ is a singleton, then $A_m = \alpha S$ where α is a scalar and we can easily take two contiuous curves as a bove joining x and y. If $V_S^+(A_m)$ is not a singleton, then there exists a non zero $u_0 \in V_S^+(A_m)$ and we work as in case (ii).

In all cases we get a continuous curve $z: [0,1] \to G$ such that $z(t)^* A_m z(t) \neq 0$ for every $t \in [0,1]$. Since the solution $\lambda_1, \lambda_2, \dots, \lambda_m$ of equation

$z(t)^* Q(\lambda) z(t) = 0$

are continuous functions of t. The roots of polynomial $z(0)^* Q(\lambda) z(0) = x^* Q(\lambda) x$ are connected to those $z(1)^* Q(\lambda) z(1) = y^* Q(\lambda) y$ by a continuous curve in $V_S^+(Q(\lambda))$. As a result, every root function λ_i is contained in a single component of $V_S^+\left(Q(\lambda)\right)$. Therefore the number of connected components in $V_S^+\left(Q(\lambda)\right)$ does not exceed m.

The following Corollary is the relation between boundedness and the number of components.

Corollary 2.11: Let $Q(\lambda)$ be as in Eq. (1) where $A_m \neq 0$ and S be a $n \times n$ Hermitian matrix. Then $V_S^+(Q(\lambda))$ has at most m connected component, if $V_S^+(Q(\lambda))$ is bounded.

The next result prove that if λ_0 is a boundary point of $V_S^+(Q(\lambda))$ then 0 is also a boundary point of $V_S^+(Q(\lambda))$.

Theorem 2.12: Let $Q(\lambda)$ be as in Eq. (1) where $A_m \neq 0$ and S be a $n \times n$ Hermitian matrix. Then 0 is a boundary point of $V_S^+(Q(\lambda_0))$ if λ_0 is a boundary point of $V_S^+(Q(\lambda))$.

Proof: By proposition 2.1 $V_S^+(Q(\lambda))$ is closed in $\mathbb{C}$ and suppose that λ_0 is on the boundary $V_S^+(Q(\lambda))$ then for any S-unit vector $x \in \mathbb{C}^n$ we have $x^* Q(\lambda) x = 0$, therefore $0 \in V_S^+(Q(\lambda_0))$ and it is sufficent to show that zero is not in an interior point $V_S^+(Q(\lambda_0))$. Let $\{\lambda_p\}_{p \in \mathbb{N}}$ be a sequence of elements in $\mathbb{C} \setminus V_S^+(Q(\lambda))$ such that $B_r(0) \subset V_S^+(Q(\lambda_0))$, we can find S-unit vectors $x_1, x_2 \in \mathbb{C}^n$ where $x_i^* S x_i = 1$ $(i = 1,2)$ such that 0 belongs to the interior of the line segment

$0 \in [x_1^* Q(\lambda) x_1, x_2^* Q(\lambda) x_2] \subset B_r(0)$

Then the line segment close to the origin for sufficiently small r. By proposition 2.1 $V_S^+(Q(\lambda))$ is closed, then the equalities

$\lim_{p \to \infty} x_i^* Q(\lambda_p) x_i = x_i^* Q(\lambda_0) x_i; \quad i = 1,2$

This implies that $0 \in V_S^+(Q(\lambda_p))$, for sufficiently large p. Thus for suitable S-unit vector $x \in \mathbb{C}^n$ we have $x^* Q(\lambda_p) x = 0$, this means which contradicts the assumption.

The following Theorem is the $V_S^+(Q(\lambda))$ is singleton if zero does not belong to the $V_S(A_m)$ and $Q(\lambda) = A_m(\lambda I - \alpha I)^m$.

Theorem 2.13: Let $Q(\lambda)$ be as in Eq. (1) where, $A_m \neq 0$ and S be a $n \times n$ Hermitian matrix. Then,

1) $V_S^+(Q(\lambda)) = \phi$ if and only if S is negative semidefinite: $V_S(Q(\lambda)) = \phi$ if and only if $S = 0$.

2) $V_S(Q(\lambda)) = \{\alpha\}$ is bounded if and only of $0 \notin V_S(A_m)$ and $Q(\lambda) = A_m(\lambda I - \alpha I)^m$ where I is an identity matrix.

The first part is obvous. Proof: part (2) Suppose that $0 \notin V_S(A_m)$ and $Q(\lambda) = A_m(\lambda I - \alpha I)^m$ .Then $V_S(Q(\lambda)) = \{\lambda \in \mathbb{C}; x^* A_m(\lambda I - \alpha I)^m x = 0, x \in \mathbb{C}^n \text{ and } x^* S x \neq 0\}$

Since $x^* A_m(\lambda I - \alpha I)^m x = (x^* A_m x)(\lambda - \alpha)^m = 0$ and given that $0 \notin V_S(A_m)$ this means that there is no $x \in \mathbb{C}^n$ such that $x^* A_m x = 0$. Hence $V_S(Q(\lambda)) = \{\alpha\}$. Conversely, suppose that $V_S(Q(\lambda)) = \{\alpha\}$. So, the only root of the equation $x^* Q(\lambda) x = 0$ is α and $x^* Q(\lambda) x = x^*(A_m \lambda^m + A_{m-1}\lambda^{m-1} + \dots + A_0) x = (x^* A_m x)(\lambda - \alpha)^m$. If $x^* A_m x = 0$ then by theorem 2.7 $V_S(Q(\lambda))$ is unbounded which is contradicts to $V_S(Q(\lambda)) = \{\alpha\}$. Hence $0 \notin V_S(A_m)$ and $Q(\lambda) = A_m(\lambda I - \alpha I)^m$.

Example 2.14: Let $Q(\lambda) = I\lambda^2 + A\lambda + B$ be a 2×2 quadratic matrix polynomial where

$A := \begin{pmatrix} 0 & 2.8i \\ -2.8i & 0 \end{pmatrix}$, $B := \begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix}$,

and let S be a 2×2 Hermitian matrix

$S := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

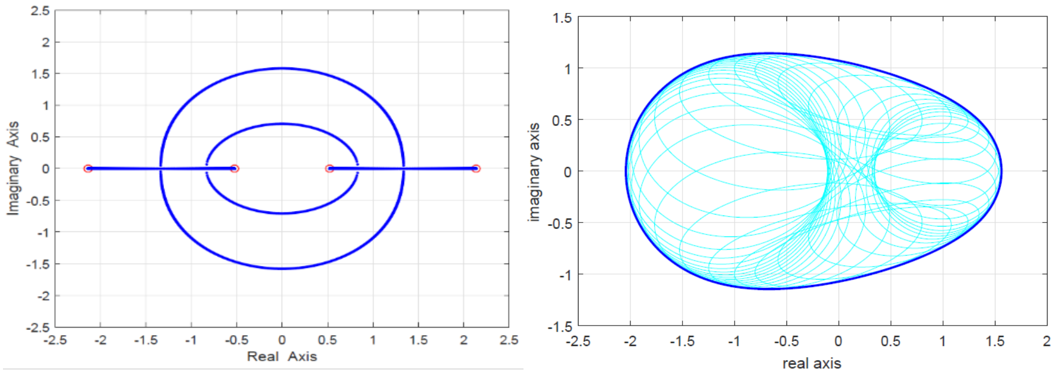

Fig. 1: The left hand side is $W(Q(\lambda))$ and the right hand side is $V_S^+(Q(\lambda))$

Remark 2.15: In this example since $Q(\lambda)$ is selfadjoint, then $W(Q(\lambda))$ is symmetric about the real axis, moreover $W(Q(\lambda))$ has four corners which are the eigenvalues of $W(Q(\lambda))$. In contrast $V_S^+(Q(\lambda))$ have no corners and the spectrum of $Q(\lambda)$ lies in the interior of $V_S^+(Q(\lambda))$.

3. Numerical range of polynomial operator matrices

In a complex Hilbert space H with an inner product $\langle .,. \rangle$, we consider a polynomial operator of degree m

$Q(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_0$ (7)

where A_j are bounded operators for $j = 0,1,\ldots,m$ with $A_m \neq 0$. A sesquilinear form $(x,y) \to \langle x,y \rangle_S := \langle Sx,y \rangle$ where, S is self-adjoint,one may define $V_S(Q(\lambda))$ as

$V_S(Q(\lambda)) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x,x \rangle = 0\colon x \in H,\ \langle x,x \rangle_S \neq 0\}$ (8)

and define $V_S^+(Q(\lambda))$ and $V_S^-(Q(\lambda))$ analogously.

Some results are the same as for the finite case, and we can get something similar to Theorems 2.9, 2.12, and 2.13. For instance, there is the following statement

Proposition 3.1: Let $Q(\lambda)$ be as in Eq. (7), $A_m \neq 0$ and S be a self-adjoint operator.

i) $V_S(Q(\lambda+\alpha)) = V_S(Q(\lambda)) - \alpha$ for any $\alpha \in \mathbb{C}$.

ii) $V_S(\alpha Q(\lambda)) = V_S(Q(\lambda))$ for any non-zero $\alpha \in \mathbb{C}$.

iii) If $\langle A_i x,x \rangle = 0$ for all i, then $V_S(Q(\lambda))$ is a hole complex plane.

iv) If $P(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m$ then $V_S(P(\lambda)) \setminus \{0\} = \{\mu^{-1} \in \mathbb{C}\colon \mu \in V_S(Q(\lambda))\}$

Proof:

i) $V_S\big(Q(\lambda+\alpha)\big) = \{\lambda \in \mathbb{C}; \langle Q(\lambda+\alpha)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

Choose $\beta = \lambda + \alpha$ then

$V_S\big(Q(\lambda+\alpha)\big) = \{\beta - \alpha \in \mathbb{C}; \langle Q(\beta)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\beta \in \mathbb{C}; \langle Q(\beta)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\} - \alpha$

$= V_S(Q(\beta)) - \alpha$

Replacing β by λ we get the result.

ii) $V_S\big(\alpha Q(\lambda)\big) = \{\lambda \in \mathbb{C}; \langle \alpha Q(\lambda)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\lambda \in \mathbb{C}; \alpha\langle Q(\lambda)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

Since $\alpha \neq 0$ hence $x^*Q(\lambda)x = 0$

$V_S\big(\alpha Q(\lambda)\big) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= V_S\big(Q(\lambda)\big)$

iii) $V_S\big(Q(\lambda)\big) = \{\lambda \in \mathbb{C}; \langle Q(\lambda)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\lambda \in \mathbb{C}; \langle (A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_0)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\lambda \in \mathbb{C}; \langle A_m x,x \rangle\lambda^m + \langle A_{m-1}x,x \rangle\lambda^{m-1} + \cdots + \langle A_0\, x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

Since $\langle A_i x,x \rangle = 0$ then

$V_S\big(Q(\lambda)\big) = \{\lambda \in \mathbb{C}, \text{for some } x \in H, \langle Sx,x \rangle = 1\} = \mathbb{C}$.

iv) $V_S\big(P(\lambda)\big) \setminus \{0\} = \{\lambda \in \mathbb{C}; \langle P(\lambda)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\lambda \in \mathbb{C}; \langle (A_o\lambda^m + A_1\lambda^{m-1} + \cdots + A_m)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

Since $0 \notin V_S\big(P(\lambda)\big)$ we can choose $\mu = 1/\lambda$ then

$V_S\big(P(\lambda)\big) \setminus \{0\} = \{1/\mu \in \mathbb{C}; \langle (A_o(1/\mu)^m + A_1(1/\mu)^{m-1} + \cdots + A_m)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\mu^{-1} \in \mathbb{C}; \langle \mu^m(A_o + A_1\mu + \cdots + A_m\mu^m)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\mu^{-1} \in \mathbb{C}; \mu^m\langle Q(\mu)x,x \rangle = 0, \text{for some } x \in H, \langle Sx,x \rangle = 1\}$

$= \{\mu^{-1} \in \mathbb{C}\colon \mu \in V_S(Q(\lambda))\}$

Remark 3.2: The same properties are holding when $V_S(Q(\lambda))$ is replaced by $V_S^{\pm}(Q(\lambda))$. Furthermore, for these sets we have additional properties.

1) If S is positive operator then $V_S(Q(\lambda)) = V_S^+(Q(\lambda))$.

2) If S is negative operator then $V_S(Q(\lambda)) = V_S^-(Q(\lambda))$.

Be careful when studying infinite dimensional operators by applying finite dimensional techniques. For example, in the case of infinite dimensions, even if S is a positive operator, the condition like (iv) in Proposition 2.1 or Theorem 2.5 may not be obtained because S^{-1} may not exist. It's clear that V-numerical range of matrix polynomial is bounded as we shown in Theorem 2.7, but in infinite dimensional $V_S\big(Q(\lambda)\big)$ is not closed. The following Theorem proved that $V_S(Q(\lambda))$ is bounded if $0 \notin \overline{V_S(Q(\lambda))}$.

Theorem 3.3: Let $Q(\lambda)$ be as in Eq. (7) and let S be a self-adjoint operator. Then $V_S^+(Q(\lambda))$ is bounded if and only of $0 \notin \overline{V_S^+(A_m)}$.

Proof: Suppose that $G = \{x \in H: \langle Sx, x\rangle = 1\}$. Let $0 \notin \overline{V_S^+(A_m)}$ then $0 \notin V_S^+(A_m)$ and $\mu = min\{|z|: z \in W_S^+(A_m)\}$, then there exists a positive real number M such that

$|\langle A_m x, x\rangle\lambda^m| \geq |\mu\lambda^m| > \sum_{k=0}^{m-1} |\langle A_k x, x\rangle|\lambda^m$

for $x \in G$ and $\lambda \in \mathbb{C}$ with $|\lambda| > M$. It clearly follows that $V_S^+(Q(\lambda)) \subseteq \{z \in \mathbb{C}: |z| \leq M\}$.Conversely, consider $V_S^+(Q(\lambda))$ is bounded and $0 \in \overline{V_S^+(A_m)}$, Let $x \in G$ and $\langle A_m x, x\rangle = 0$. Since $V_S^+(Q(\lambda))$ is bounded, then there must be at least one coefficient $A_r (r \neq m)$ such that $\langle A_r x, x\rangle \neq 0$. A sequence of unitary operator $\{U_k\}_{k\in\mathbb{N}}$ converging to the identity operator I, such that $\langle U_k^* A_r x, U_k x\rangle \neq 0$ and $U_k^* S U_k = S$ for $k \in \mathbb{N}$. Then $x_k = U_k x \to x$ and $\langle S x_k, x_k\rangle = 1$ for $k \in \mathbb{N}$. For a fixed $\epsilon > 0$ and for all sufficiently large k it is obvious that $\langle A_r x_k, x_k\rangle > \epsilon$. Since $W_S^+(Q(\lambda))$ is bounded, the function $\frac{\langle A_r x_k, x_k\rangle}{\langle A_m x_k, x_k\rangle}$ of the polynomial $\langle Q(\lambda) x_k, x_k\rangle$ is also bounded for all $k \in \mathbb{N}$, which is contradiction.

4. Conclusions

This paper illustrate the location of $V_S(Q(\lambda))$ the complex plane and a theorem concerning the boundary of $V_S(Q(\lambda))$ is also obtained. On the other hand describe possible generalazations of those results including their extensions to bounded linerar operators on an infinite dimensional Hilbert space .

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تعميم المدى العددي لمصفوفات متعددة الحدود المؤثرة

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الملخص

نفـرض ان $Q(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0$ هـو مصـفوفة مـؤثرة متعـددة الحـدود و حيـث ان $A_i \in M_n(\mathbb{C})$ لكـل $i = 0,1,\ldots,m$ هي $n \times n$ مصفوفة عقدية لتكن λ متغير عقدي ل $n \times n$ مصفوفة هيرميشن S. نعرف V مدى العددي لمصفوفة متعددة الحدود ل $Q(\lambda)$ مثل

$$V_S(Q(\lambda)) = \{ \lambda \in \mathbb{C};\ \langle Q(\lambda)x, x\rangle = 0,\ \text{ for some } x \in \mathbb{C}^n, \langle x, x\rangle_S \neq 0 \}$$

حيث $\langle x, y\rangle_S = x^* S y$. في هذه بحث تم دراسة $V_S(Q(\lambda))$ و ركزنا على الخواص الهندسية ل $V_S(Q(\lambda))$ نحن نأخذ الاعتبار موقع $V_S(Q(\lambda))$ في المستوى العقدي و تم ايضا الحصول على مبرهنة تتعلق حدود $V_S(Q(\lambda))$. تم وصف التعميم لنتائجنا بما في ذالك توسعات المؤثرات الحدودية الخطية في الأبعاد الانهائية لفضاء هلبرت.