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Comparative Numerical Solution of Fractional Spline with Continuity Equations

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ABSTRACT

In this paper, constructed a fractional polynomial spline to compute the solution of FDEs; the spline interpolation with fractional polynomial coefficients must be constructed using the Caputo fractional derivative. For the provided spline function, error bounds were studied and a stability analysis was completed. To consider the numerical explanation for the provided method and compared, three examples were studied. The fractional spline function, which interpolates data, appears to be useful and accurate in solving unique problems, according to the research.

1. Introduction

Generalized differential equations (GDEs) are equations in which fractional order derivatives replace integral order derivatives. Ordinary differential equations (ODEs) are a type of differential equation used to describe dynamic events in physics, biology, and chemistry, among other fields. On the other hand, some complicated systems cannot be solved by simple differential equations. As a result, condition models are improved using FDEs instead of integer order FDEs, [1, 2, 3] are examples of this. FDEs, on the other hand, are too challenging to study using analytical methods, and there is no theoretical foundation for this subject, Mathematicians have recently found new numerical solution methods as a result of all this. FDEs can be solved using fractional order differential equations in a range of domains, including engineering, chemistry, and physics [4]. A simple method must be used to solve some equations. Many known models are defined as none (fractional) order derivatives in diffusion processes, viscoelasticity, electrochemistry, and other fields. Some numerical algorithms for solving various derivatives of fractional order problems are [5] and [6]. One of the most useful techniques for the numerical approximation of functions is fractional order spline functions. According to the researchers, other difficult [6] and [7] problems should be replaced by new work and projects. On one hand, there is a great method for the numerical approximation of functions using spline functions. Researchers, on the other hand, can come up with some new, exciting, and difficult problems to solve [7]. For example [2], Spline lacunary interpolation develops when a related problem involving a function and its derivative [7, 8] and [9, 10] arises. Spline interpolation, also known as fractional spline interpolation, is discussed in this paper, as well as error estimation and convergence analysis for the fractional spline function. To find a new form of class $C^{\frac{7}{2}}$- An approximation method using the lacunary spline is built and applied to a numerical solution to a fractional initial value problem. The current article can be considered to repeat the formulated spline polynomial in the first section, the error estimations in the second section, and stability and convergence analysis in the third section. The fourth section, which presents the collations of our numerical results with exact solutions, has been studied. This section compared and shows the results and conclusions of the study, See [11, 13].

2. Mathematical Preliminaries and Basic Definitions

This section will go over the many definitions of the fractional derivative as well as Taylor's Theorem, which we used in our work. To define, different methods are used. The most common fractional derivatives are the Riemann-Liouville and Caputo derivatives.

Definition 2.1: [12] (Caputo Fractional Derivative) The Caputo derivative operator of order α is defined as:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \left(\frac{d}{du}\right)^n f(u)du, n = \lceil\alpha\rceil and \alpha > 0. $$

For $a = 0$, we introduce the notation:

$$ {}^C D_t^\alpha f(t) = D^\alpha f(t) $$

Definition 2.2: [12] (Fractional Derivative of Order) The Riemann-Liouville derivative of order α can be defined as:

$$ {}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\alpha-1} f(u)du. $$

For every α, and $n = \lceil\alpha\rceil$

Definition 2.3:[11] Suppose that $D_a^{z\lambda} G(x) \in \mathbb{C}[a,b]$ for $z = 0,1,\dots,n+1$ and $0 < \lambda \le 1$ then we have the Taylor series expansion about $x = \tau$

$$ g(x) = \sum_{k=0}^{n} \frac{(x-\tau)^{k\lambda}}{\Gamma(k\lambda+1)} D_a^{k\lambda} g(\tau) + \frac{\left(D_a^{(n+1)\lambda} g\right)(\xi)}{\Gamma((n+1)\lambda+1)} (x-\tau)^{(n+1)\lambda} \text{ with } a \le \xi \le x, $$

for all $x \in [a,b]$ where $D_a^{z\lambda} = D_a^\lambda \cdot D_a^\lambda \dots D_a^\lambda$ (z times).

3. Theoretical of the Spline Method

Construct the approximate solution of FDEs in this part to use a spline function with fractional polynomial coefficients. Stability and error bounds should be considered in relation to the following theorems.

Theorem 3.1:

There exist a unique spline $S(x) \in S_{\left(n,\frac{7}{2}\right)}$, given the real numbers $D^{\left(\frac{3}{2}\right)} s_j, j = 0,1,\dots,N, s_0, D^{\left(\frac{1}{2}\right)} s_0$ and Ds_0 Such that:

$$ s(x_j) = y_j $$

$$ D^{(1/2)} s(x_j) = y_j^{\left(\frac{1}{2}\right)}, \; D^{(1/2)} s(x_{j+1}) = y_{j+1}^{\left(\frac{1}{2}\right)} $$

$$ D^{(3/2)} s(x_{j+1}) = y_{j+1}^{\left(\frac{3}{2}\right)} \quad \dots .(1) $$

$$ D^{(\alpha)} s(x_j) = D^{(\alpha)} f(x_j) \text{ , where } = 0,1,\dots,n \text{ . } \alpha = \frac{1}{2} $$

Proof:

Consider the spline method in [10], when we created and introduced a new boundary condition for the spline function $s(x) \in S\left(n,\frac{7}{2}\right)$ in the interval (0,1].

$$ s(x) = A(x)S_i + (x-x_i)^{\frac{1}{2}} \left[B(x)S_i^{\left(\frac{1}{2}\right)} + C(x)S_{i+1}^{\left(\frac{1}{2}\right)}\right] + D(x)(x-x_i)' S_i' + (x-x_i)^{\frac{3}{2}} \left[E(x)S_i^{\left(\frac{3}{2}\right)} + F(x)S_{i+1}^{\left(\frac{3}{2}\right)}\right] + G(x)(x-x_i)^2 S_{i+1}^{(2)} \quad \dots(2) $$

We get the following by using all of the conditions in equation (1) and simplified them.

$$ A(x) = 1 $$

$$ B(x) = \frac{2x^{\frac{1}{2}}}{\pi^{(1/2)}} + \left(\frac{-45\cdot\pi^{\left(\frac{3}{2}\right)} + 120\cdot\pi^{\left(\frac{1}{2}\right)}}{42\pi-128}\right)x^2 + \left(\frac{16\cdot\pi^{\left(\frac{1}{2}\right)}}{21\pi-64}\right)x^{\frac{5}{2}} + \left(\frac{15\cdot\pi^{(3/2)} - 60\cdot\pi^{(1/2)}}{42\pi-128}\right)x^3 $$

$$ C(x) = \left(\frac{45\cdot\pi^{\left(\frac{3}{2}\right)} - 120\cdot\pi^{\left(\frac{1}{2}\right)}}{42\pi-128}\right)x^2 + \left(\frac{-16\cdot\pi^{\left(\frac{1}{2}\right)}}{21\pi-64}\right)x^{\frac{5}{2}} + \left(\frac{-15\cdot\pi^{(3/2)} + 60\cdot\pi^{(1/2)}}{42\pi-128}\right)x^3 $$

$$ D(x) = x + \left(\frac{-45\pi+120}{21\pi-64}\right)x^2 + \left(\frac{32}{21\pi-64}\right)x^{\frac{5}{2}} + \left(\frac{15\pi-60}{21\pi-64}\right)x^3 \quad \dots(3) $$

$$ E(x) = \frac{4x^{\frac{3}{2}}}{3\cdot\pi^{(1/2)}} + \left(\frac{-45\cdot\pi^{\left(\frac{3}{2}\right)} + 144\cdot\pi^{\left(\frac{1}{2}\right)}}{84\pi-256}\right)x^2 + \left(\frac{-16\cdot\pi^{\left(\frac{1}{2}\right)}}{105\pi-320}\right)x^{\frac{5}{2}} + \left(\frac{15\cdot\pi^{(3/2)} - 40\cdot\pi^{(1/2)}}{84\pi-256}\right)x^3 $$

$$ F(x) = \left(\frac{-45\cdot\pi^{\left(\frac{3}{2}\right)} + 96\cdot\pi^{\left(\frac{1}{2}\right)}}{84\pi-256}\right)x^2 + \left(\frac{96\cdot\pi^{\left(\frac{1}{2}\right)}}{105\pi-320}\right)x^{\frac{5}{2}} + \left(\frac{15\cdot\pi^{(3/2)} - 80\cdot\pi^{(1/2)}}{84\pi-256}\right)x^3 $$

$$ G(x) = \frac{3\pi x^2}{21\pi-64} - \frac{256x^{\frac{5}{2}}}{315\pi-960} + \frac{5\pi x^3}{42\pi-128} $$

Evaluating the step size of equation (3), $x_{j+1} = x_j + k\lambda h$,$0 \le k \le 1$, with $s(x)$ in $[x_{j-1}, x_j]$. Since $s(x) \in C^{\frac{7}{2}}$, and the fractional continuity equations.

$s(x_j^+) = s(x_j^-)$ to $D^{\left(\frac{5}{2}\right)} s(x_j^+) = D^{\left(\frac{5}{2}\right)} s(x_j^-)$ respectively, for $j = 0,1,2,\dots,N$, leads to the equations in the following linear system:

$$ S_i = \left[1 - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon\right] S_{i-1} + \left[h^{\frac{1}{2}}\alpha + h^{\frac{1}{2}}\beta\right] S_{i-1}^{\left(\frac{1}{2}\right)} + \left[\frac{2h\beta + \sqrt{\pi}h\gamma}{\sqrt{\pi}}\right] S_{i-1}' + \left[\frac{4h^2\beta + 6h^2\varepsilon + 3\sqrt{\pi}h^2\zeta}{3\sqrt{\pi}}\right] S_{i-1}^{(2)} \quad \dots(4) $$

$$ h^{\frac{1}{2}} S_i^{\left(\frac{1}{2}\right)} = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_1 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_1}{\beta_1}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_1}{\beta_1}\right] S_{i-1}^{\left(\frac{1}{2}\right)} - \left[\frac{h\gamma_1}{\beta_1}\right] S_{i-1}' - \left[\frac{2h^2\varepsilon_1 + \sqrt{\pi}h^2\zeta_1}{\sqrt{\pi}\beta_1}\right] S_{i-1}^{(2)} \quad \dots(5) $$

$$ h' S_i' = -\left[\frac{\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_2 - 2\gamma_2 h^{\frac{3}{2}}\lambda^{\frac{3}{2}}}{\gamma_2\sqrt{\pi}}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_2 + h^{\frac{1}{2}}\beta_2}{\gamma_2}\right] S_{i-1}^{\left(\frac{1}{2}\right)} - \left[\frac{2h\beta_2}{\gamma_2\sqrt{\pi}}\right] S_{i-1}' - \left[\frac{4h^2\beta_2 + 6h^2\varepsilon_2 + 3\sqrt{\pi}h^2\zeta_2 - 3\sqrt{\pi}h^2\gamma_2}{\gamma_2\sqrt{\pi}}\right] S_{i-1}^{(2)} \quad \dots .(6) $$

$$h^2 S_i^{(2)} = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_4}{\zeta_4}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_4 + h^{\frac{1}{2}}\beta_4}{\zeta_4}\right] S_{i-1}^{(\frac{1}{2})} - \left[\frac{2h\beta_4+\sqrt{\pi}h\gamma_4}{\zeta_4\sqrt{\pi}}\right] S'_{i-1} - \left[\frac{4h^2\beta_4+6h^2\varepsilon_4}{3\zeta_4\sqrt{\pi}}\right] S_{i-1}^{(2)} \quad \ldots(7)$$

The theorem's proof is now complete.

Hint: Using the above theorem, we can show that the spline method's model exists and is unique, since it's easily to show that algebraically, after change to linear system. However, the following theorems can be used to show that the spline method construction's convergence analysis is correct.

Theorem 3.2: Let $f \in C^4[0,h]$ be the exact function, and $s(x)$ be a single fractional polynomial that exists for all $t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ for all points in [a, b], and that matches $f(x)$ and its first n-1 derivatives $f^{(r)}$at 0 and h. Then

$$\left|s^{(t)}(x) - f^{(t)}(x)\right| \leq \frac{h^{t+a}}{(2m-2)!}\|f^{(4)}\|$$

Where

$\left|e^{(t)}\right| = \left|f^{(t)}(x) - S^{(t)}(x)\right|$ and a be a constant, $K = \max_{0\leq x\leq h}\left|f^{(2m)}(x)\right|$

Proof:

Since $D^{\frac{1}{2}}s(x)$ is Hermite interpolation polynomial of degree 3, and matching $D^{\frac{1}{2}}f(x), D^{\frac{3}{2}}f(x)$, at $x = x_j, x_{j+1}$ so for any $x \in [x_j, x_{j+1}]$, using [15, 16], and let

$m = 2$, $g = f^{(\frac{1}{2})}$ and $p_3 = s^{(\frac{3}{2})}(x)$, we get:

$\left|s^{(\frac{1}{2})} - f^{(\frac{1}{2})}\right| \leq \frac{h^5}{2!}D^{(4)}f$, also if we put $g = D^{\frac{3}{2}}f(x)$ and $P_3 = D^{\frac{3}{2}}s(x)$ we get

$\left|s^{(\frac{3}{2})} - f^{(\frac{3}{2})}\right| \leq \frac{h^3}{2!}D^{(4)}$ then we can get $|s(x) - s(0) + f(0) - f(x)| \leq \frac{h^6}{2!}\parallel f^{(4)}\parallel$

Since $s(0) = f(0)$ and $x \in [0,1]$ then the last equation becomes $\left|s^{(\frac{5}{2})} - f^{(\frac{5}{2})}\right| \leq \frac{h^{\frac{7}{2}}}{2!}\parallel f^{(4)}\parallel$ and since $f^{(p)}(0) = 0, p = 1,2,$ [19] we have the following:

$$\left|s^{(\frac{1}{2})} - f^{(\frac{1}{2})}\right| \leq \frac{h^5}{2!}\parallel f^{(4)}\parallel$$

$$\left|s^{(\frac{3}{2})} - f^{(\frac{3}{2})}\right| \leq \frac{h^3}{2!}\parallel f^{(4)}\parallel$$

$$\left|s^{(\frac{5}{2})} - f^{(\frac{5}{2})}\right| \leq \frac{h^{\frac{7}{2}}}{2!}\parallel f^{(4)}\parallel .$$

4 Stability Analysis

In this section, we'll look at the stability analysis of the proposed spline method. Equations (4), (5), (6), and (7) are considered for evaluating the equation's stability and providing a method to test it, using fractional differential equations, and is dedicated to the stability of the linear system by removing the effects of errors.

$D^{(\frac{3}{2})}y(x) = -\lambda^{\frac{3}{2}}y(x), \lambda \in \mathbb{R}, y(x_0) = y_0, y'(x_0) = y'$ ... (8)

$$S_i = \left[1 - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon\right] S_{i-1} + \left[h^{\frac{1}{2}}\alpha + h^{\frac{1}{2}}\beta\right] S_{i-1}^{(\frac{1}{2})} + \left[\frac{2h\beta+\sqrt{\pi}h\gamma}{\sqrt{\pi}}\right] S'_{i-1} + \left[\frac{4h^2\beta+6h^2\varepsilon+3\sqrt{\pi}h^2\zeta}{3\sqrt{\pi}}\right] S_{i-1}^{(2)};$$

$$h^{\frac{1}{2}}S_i^{(\frac{1}{2})} = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_1 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_1}{\beta_1}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_1}{\beta_1}\right] S_{i-1}^{(\frac{1}{2})} - \left[\frac{h\gamma_1}{\beta_1}\right] S'_{i-1} - \left[\frac{2h^2\varepsilon_1+\sqrt{\pi}h^2\zeta_1}{\sqrt{\pi}\beta_1}\right] S_{i-1}^{(2)};$$

$$h'S'_i = -\left[\frac{\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_2 - 2\gamma_2 h^{\frac{3}{2}}\lambda^{\frac{3}{2}}}{\gamma_2\sqrt{\pi}}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_2 + h^{\frac{1}{2}}\beta_2}{\gamma_2}\right] S_{i-1}^{(\frac{1}{2})} - \left[\frac{2h\beta_2}{\gamma_2\sqrt{\pi}}\right] S'_{i-1} - \left[\frac{4h^2\beta_2+6h^2\varepsilon_2+3\sqrt{\pi}h^2\zeta_2-3\sqrt{\pi}h^2\gamma_2}{\gamma_2\sqrt{\pi}}\right] S_{i-1}^{(2)};$$

$$h^2 S_i^{(2)} = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_4}{\zeta_4}\right] S_{i-1} - \left[\frac{h^{\frac{1}{2}}\alpha_4 + h^{\frac{1}{2}}\beta_4}{\zeta_4}\right] S_{i-1}^{(\frac{1}{2})} - \left[\frac{2h\beta_4+\sqrt{\pi}h\gamma_4}{\zeta_4\sqrt{\pi}}\right] S'_{i-1} - \left[\frac{4h^2\beta_4+6h^2\varepsilon_4}{3\zeta_4\sqrt{\pi}}\right] S_{i-1}^{(2)};$$

This linear system can be written as follows:

$\mathrm{S_j} = DS_{i-1}, j = 1, \ldots, \mathbb{N}$,

where, $S_j = \begin{bmatrix} S_j \\ D^{(\frac{1}{2})}S_j \\ DS_j \\ D^{(2)}S_j \end{bmatrix}, S_{j-1} = \begin{bmatrix} S_{j-1} \\ D^{(\frac{1}{2})}S_{j-1} \\ DS_{j-1} \\ D^{(2)}S_{j-1} \end{bmatrix}$, and

We get the following matrix

$$T = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix} \quad \ldots(9)$$

Where:

$a_0 = \left[1 - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta - h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon\right]$, $a_1 = \left[h^{\frac{1}{2}}\alpha + h^{\frac{1}{2}}\beta\right]$, $a_2 = \left[\frac{2h\beta+\sqrt{\pi}h\gamma}{\sqrt{\pi}}\right]$ and $a_3 = \left[\frac{4h^2\beta+6h^2\varepsilon+3\sqrt{\pi}h^2\zeta}{3\sqrt{\pi}}\right]$

$b_0 = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_1 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_1}{\beta_1}\right]$, $b_1 = -\left[\frac{h^{\frac{1}{2}}\alpha_1}{\beta_1}\right]$, $b_2 = -\left[\frac{h\gamma_1}{\beta_1}\right]$ and $b_3 = -\left[\frac{2h^2\varepsilon_1+\sqrt{\pi}h^2\zeta_1}{\sqrt{\pi}\beta_1}\right]$

$c_0 = -\left[\frac{\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_2 + \sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_2 - 2\gamma_2 h^{\frac{3}{2}}\lambda^{\frac{3}{2}}}{\gamma_2\sqrt{\pi}}\right]$,$c_1 = -\left[\frac{h^{\frac{1}{2}}\alpha_2 + h^{\frac{1}{2}}\beta_2}{\gamma_2}\right]$,$c_2 = -\left[\frac{2h\beta_2}{\gamma_2\sqrt{\pi}}\right]$ and $c_3 = -\left[\frac{4h^2\beta_2+6h^2\varepsilon_2+3\sqrt{\pi}h^2\zeta_2-3\sqrt{\pi}h^2\gamma_2}{\gamma_2\sqrt{\pi}}\right]$

$d_0 = \left[\frac{h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\beta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\delta_4 + h^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varepsilon_4}{\zeta_4}\right]$, $d_1 = -\left[\frac{h^{\frac{1}{2}}\alpha_4 + h^{\frac{1}{2}}\beta_4}{\zeta_4}\right]$

$d_2 = -\left[\frac{2h\beta_4+\sqrt{\pi}h\gamma_4}{\zeta_4\sqrt{\pi}}\right]$ and

$d_3 = -\left[\frac{4h^2\beta_4+6h^2\varepsilon_4}{3\zeta_4\sqrt{\pi}}\right]$

Theorem 4.1: Equations (9) by using fractional spline method is stable.

Proof: Assume T is a complex conjugate matrix and that $|\lambda_i| \le 1$ is true. The characteristic equation will be stable as the characteristic polynomial [17] if all complex eigenvalues have negative real components.

Theorem 4.2: The matrix $(I-T)$ is invertible if T is a $n \times n$ Matrix with$\| T \|_\infty < 1$ in addition to $\|(I-T)^{-1}\|_\infty \le \frac{1}{1-\|T\|_\infty}$.

Proof: see [18].

Theorem 4.3: T is non-singular if it has n independent columns, T^{-1}exists, and $Tu = f$ has a unique solution u.

Proof: Since we have a matrix T from the linear system of equation (10), if $|T| \ne 0$, then T^{-1} exists, and the system is unique using the [17].

Theorem 4.4: Convergence is a necessary and sufficient condition if the eigenvalues satisfy the matrix T.

Proof: Assume that all the eigenvalues of T are unique, and that (theorem 3.1 [18]) the spectral radius of the matrix T is less than one. $|\lambda_i| < 1$, then the matrix T is converges.

5. Numerical illustrations

In the three numerical examples in this section, the method is used to complete all computations. Three fractional initial value problems are considered to define the class $C^{\frac{7}{2}}$ of fractional interpolation spline and to test the computational applicability of the provided method. The application of the results in two parts shows the value of the proposed technique. Tables 1 , 2 and 3 are given below.

The term $e, e^{\left(\frac{1}{2}\right)}$ and $e^{(1)}$ represent the maximum magnitude errors $|e(x)| = |s(x) - y(x)|, \left|D^{\left(\frac{1}{2}\right)}e(x)\right| = \left|D^{\left(\frac{1}{2}\right)}s(x) - D^{\left(\frac{1}{2}\right)}y(x)\right|$, and $|e'(x)| = |S'(x) - y'(x)|$ respectively.

Example 1: [13] Consider the fractional differential equation

$D^2y(t) + 2D^\alpha y(t) + y(t) = 2t + \frac{4}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{1}{3}t^3$, $0 < \alpha < 1$, subject to $y(0) = y'(0) = 0$. it is easily verified that the exact solution to this problem is $y(x) = \frac{1}{3}t^3$

Table 1: Absolute error of $S(\mathrm{x})$ and its derivative of example 1

h	$\lvert s(x) - f(x)\rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)}\right\rvert$	$\lvert s' - f'\rvert$	Exact solution	Approximation solution
0.001	2.1465×10^{-11}	6.5181×10^{-8}	2.6172×10^{-1}	3.3333×10^{-10}	3.5479×10^{-10}
0.05	1.6201×10^{-5}	2.9130×10^{-4}	1.4913×10^{-1}	4.1666×10^{-5}	5.7867×10^{-5}
0.02	7.4044×10^{-7}	2.2475×10^{-6}	2.1817×10^{-1}	2.6666×10^{-6}	3.4071×10^{-6}
0.1	1.2287×10^{-4}	5.7175×10^{-3}	3.4151×10^{-2}	3.3333×10^{-4}	4.5620×10^{-4}

Table 2: In comparison to the method of [13], the absolute error in Example 1 is shown.

h	Our method	Ref[13]
0.001	2.1465×10^{-11}	5.4910×10^{-8}
0.01	6.7043×10^{-8}	5.4910×10^{-5}
0.1	1.2287×10^{-4}	5.4910×10^{-2}

Example 2: [11] Consider the boundary value problem for the fractional differential equation

$y^{(4)}(t) + 0.05D^\alpha y(t) = g(t),\ t \in (0,1), y(0) = 0, y(1) = 0, y^{(2)}(0) = 0, y^{(2)}(1) = 8$

With exact solution $y(t) = t^5 - t^4$. Here,$g(t) = 120t - 24 + \frac{6}{\Gamma(6-\alpha)}t^{5-\alpha} - \frac{1.2}{\Gamma(5-\alpha)}t^{4-\alpha}$.

Table 3: Absolute error of $S(\mathrm{x})$ and its derivative of example 2

h	$\lvert s(x) - f(x)\rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)}\right\rvert$	$\lvert s' - f'\rvert$	Exact solution	Approximation solution
0.001	1.3307×10^{-16}	3.3889×10^{-10}	1.5677×10^{-3}	-9.99×10^{-13}	-9.9886×10^{-13}
0.05	9.4574×10^{-7}	2.6324×10^{-5}	4.0790×10^{-2}	-5.9375×10^{-6}	-4.9917×10^{-6}
0.02	4.3991×10^{-9}	2.9125×10^{-6}	2.5292×10^{-2}	-1.568×10^{-7}	-1.5240×10^{-7}
0.1	5.4190×10^{-5}	1.0408×10^{-3}	1.5526×10^{-2}	-9×10^{-5}	-3.5809×10^{-5}

Table 4: In comparison to the method of [11], the absolute error in Example 2 is shown.

h	Our method	Ref[11]
0.001	1.3307×10^{-16}	3.2163×10^{-13}
0.02	4.3991×10^{-9}	6.3200×10^{-8}
0.01	7.6759×10^{-11}	1.7824×10^{-10}

Example 3: [20] Consider the fraction boundary value problem:

$$\frac{1}{6}y''(x) + \frac{24}{105}D^{(1/2)}y(x) + y(x) = x(x^2+1) + \frac{4}{\sqrt{\pi}}x^2\sqrt{x}$$

$y(0) = 0, y(0.5) = 0.125$, the exact solution of equation is $y(x) = x^3$

Table 5: Absolute error of $S(x)$ and its derivative of example 3

h	$\lvert s(x)-f(x)\rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)}-f^{\left(\frac{1}{2}\right)}\right\rvert$	$\lvert s'-f'\rvert$	Exact solution	Approximation solution
0.001	6.4395×10^{-11}	1.9554×10^{-7}	7.8516×10^{-1}	1×10^{-9}	1.0643×10^{-9}
0.05	4.8603×10^{-5}	8.7392×10^{-4}	4.4741×10^{-1}	1.2500×10^{-4}	1.7360×10^{-4}
0.02	2.2213×10^{-6}	6.7427×10^{-6}	6.5451×10^{-1}	8×10^{-6}	1.0221×10^{-5}
0.1	3.6861×10^{-4}	1.7152×10^{-2}	1.0245×10^{-1}	1×10^{-3}	1.3686×10^{-3}

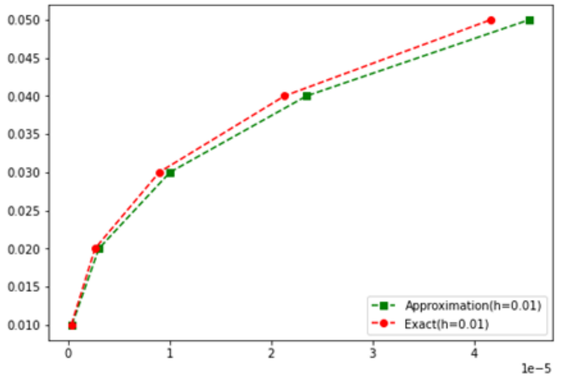

Fig. 1: For example 1, comparing the solution obtained by the spline method with the exact solution of h=0.01

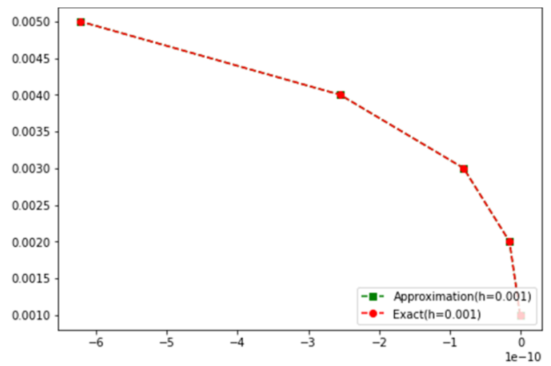

Fig. 2: For example 2, comparing the solution obtained by the spline method with the exact solution of h=0.001

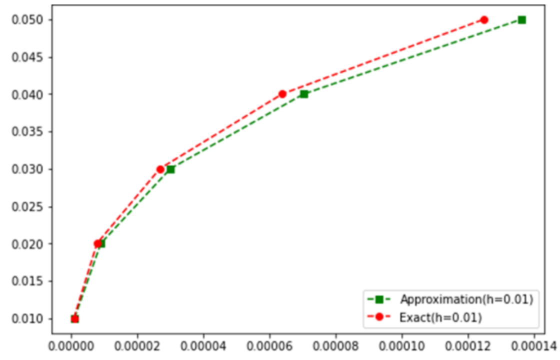

Fig. 3: For example 3, comparing the solution obtained by the spline method with the exact solution of h=0.01

6 Conclusions: In this paper, we presented an approximation solution for fractional differential equations. This scheme presented in this work is novel and used to calculate the Caputo derivative. The three examples illustrate that a given method can successfully approximate the answer; in comparison to those methods already spline method developed than the results [11, 13]; on the other hand, at least h should be used as a step size. The proposed method approximates the higher-order derivative s while concurrently approximating the solution of fractional initial value problems. A spline method is also developed based on the coefficient and convergence of the approach.

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الحل العددي المقارن لسبلاين الكسرية مع معادلات الاستمرارية

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الملخص

في هذا البحث ، تم إنشاء متعددة حدود سبلاين الكسرية لأيجاد حل المعادلات التفاضلية الكسرية ؛ يجب أن يتم إنشاء استيفاء سبلاين بمعاملات كثيرة الحدود الكسرية باستخدام مشتق كابوتو(Caputo) الكسرية. بالنسبة لدالة سبلاين المقدمة ، تمت دراسة حدود الخطأ وأنجز تحليل الثبات . للنظر في التفسير العددي للطريقة المقدمة ومقارنتها ، تمت دراسة ثلاثة أمثلة. وفقًا للبحث ظهر أن دالة سبلاين الكسرية ، التي تقحم البيانات ، مفيدة ودقيقة في حل المسائل الوحيدة .