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SOME RESULTS ON STRONGLY π-REGULAR RIN

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ABSTRACT

In this paper we study the strongly $\boldsymbol{\pi}$- regular ring (for short st. $\boldsymbol{\pi}$-reg. rg.) and some properties also give some new results of st. π-reg. rg. and its connection with other rings.

1. Introduction

Let R be a ring we study concept of st. $\boldsymbol{\pi}$-reg. ring., which introduced 1954 by Azumaya [2], and we give background theorems and corollary which we need in this paper also give some new results of st. π-reg. rg. and its connection with other rg.s. An element $a \in \mathcal{R}$ is called regular element if there exists some $b \in \mathcal{R}$ such that $aba = a$. A ring is called regular ring if every element is regular.

2. Strongly $\boldsymbol{\pi}$-reg. ring.

Definition 2.1 [2]:

we call a st. $\pi - regular$ if it is both right $\pi - regular$ and left $\pi - regular$.

Now it can readily be seen that a power a^n of a is right (or left) reg. iff here exists an element b of s.t. $a^{n+1}b = a^n$ $(or\ ba^{n+1} = a^n)$, where a, b ∈R.

Theorem 2.2 [2]:

Under the assumption that $\mathcal{R}$ is of bounded index, the following four conditions are equivalent to ∀ other:

(1) $\mathcal{R}$ π-reg.,
(2) $\mathcal{R}$ is right .π-reg.,
(3) $\mathcal{R}$ is left π-reg.,
(4) $\mathcal{R}$ is st. π-reg..

Lemma 2.3 :

Let b, c satisfy $a^{n+1}b = a^n, ca^{m+1} = a^m$ for some $n, m \in Z$. Then they satisfy $a^{m+1}b = a^m, ca^{n+1} = a^n$ too.

Proof: When $m \geq n$ $a^{m+1} = a^m$ follows immediately from $a^{n+1}b = a^n$. Suppose now $m < n$. Then $a^m = ca^{m+1}$ implies
$a^m (= c^2 a^{m+2} = \cdots) = c^{n-m}a^n$, and so we obtain
$a^{m+1}b = c^{n-m}a^{n+1}b = c^{n-m}a^n = a^m$.
Similarly , we can verify the validity of $ca^{n+1} = a^n$. ■

Proposition 2.4[2] :

Every St. π-reg. element is π-reg..

Proposition 2.5 [5]:

Let $\mathcal{R}$ be a st. π-reg. ring. Then for all $a \in \mathcal{R}$, there exists a positive integer n s.t. $a^n = eu = ue$ for some $e \in Id(\mathcal{R})$ and some $u \in U(\mathcal{R})$, where $Id(\mathcal{R})$ and $U(\mathcal{R})$ denote the set of idempotent of $\mathcal{R}$ and the set of units of $\mathcal{R}$, respectively.

Definition 2.6 [8]:

A central idempotent in A is an idempotent in the central of A.

Theorem 2.7 [5]:

Let $\mathcal{R}$ be a rg. with central idempotent. Then $\mathcal{R}$ is st. π-reg. iff $N(\mathcal{R}) = J(\mathcal{R})$ and $\mathcal{R}/N(\mathcal{R})$ is reg., where $N(\mathcal{R}), J(\text{R})$ denoted the set of all nilpotent and the Jacobson of $\mathcal{R}$ respectively.

Definition 2.8 [1]:

A ring $\mathcal{R}$ is called an exchange ring if for every $a \in \mathcal{R}$, there exists $e \in Id(\mathcal{R})$ such that $e \in a\mathcal{R}$ and $1 - e \in (1 - a)\mathcal{R}.$ $(Id(\mathcal{R})$ meanls the set of all idempotent in $\mathcal{R}$).

Remark 2.9 [1]:

Every st. π-reg. rg. is an exchange rg..

Theorem 2. 10[1]:

Let $\mathcal{R}$ be an exchange ring and let a be a reg. element of is st. π-reg., then a is unit-reg element of A..

Definition 2.11 [4]:

Let I be an ideal of a $ring\ \mathcal{R}$. We say that I is a st. π-reg. ideal of $\mathcal{R}$ in case for any $a \in I$ if there exist $n \in N$ and $b \in I$ s.t.
$a^n = a^{n+1}b.$

Theorem 2.12 [4]:

Let I be an ideal of a rg. $\mathcal{R}$. Then the following are equivalent**:**

(1) I is st. π-reg.**.**

(2) Every element in I is st. π-reg. element.

Proposition 2.13 [6]:

Every right (or left) π-reg. rg. $\mathcal{R}$ is st. π-reg..

Remark 2.14:

The factor ring of the integers with respect to the ideal generated by the integer 4 is a st. π-reg. rg. which is not a reg. rg..

Theorem 2.15 [7]:

Let $\mathcal{R}$ be a rg. and I an ideal of $\mathcal{R}$.

(1)If $\mathcal{R}$ is a st. π-reg. rg. then so is ${}^{\mathcal{R}}/_{I}$ is st. $\pi - regular\ ring$.

(2)Assume that I is a reg. ideal of $\mathcal{R}$. Then , $\mathcal{R}$ is a st. π-reg. rg. Iffso is ${}^{\mathcal{R}}/_{I}$.

Proposition 2.16 [7]:

Let $\mathcal{R}$ be a rg. and P be a prime ideal of $\mathcal{R}$. If $R\ /\ p$ is st. π-reg., then so is $\mathcal{R}_P$.

Definition 2.17:

Let $\mathcal{R}$ be a rg. and let $a \in \mathcal{R}$, the element a is called w-idempotent if for some positive integer n, a^n is an idempotent, i.e. $(a^n)^2=a^n$.

Remark: The property that a is an w-idempotent is equivalent to the property that ∃distinct positive integer n, m s.t. $a^n = a^m$.

On the other hand if there exists positive integer n, m with $n > m$ with $a^n = a^m$. Then there is some $t > 0$ s.t. $t(n - m) >$ m.

Let $k = t(n - m) = m$ and let $f = a^{m+k} = a^{t(n-m)}$ then

$a^m = a^n = a^m \cdot a^n a^{-m} = a^m a^{t(n-m)}$

Thus

$f = a^{t(n-m)} = a^{\boldsymbol{m+k}} = a^k \cdot a^m = a^k a^m a^{t(n-m)} = a^k a^m a^{k+m} = f^2$

∴ a is w-idempotent.

Theorem 2.18:

Let a be a st. reg. element of a ring R. There exists one and only one element c s.t. $ac = ca$, $a^2c = (ca^2) = a$ and $ac^2 (= c^2a) = c$, and in particular a is reg. element. For any element b s.t. $a^2b = a$, c coincides with ab^2. Moreover, c commutes with every element which is commutative with a.

Proof: Let b, d be two elements s.t. $a^2b = a, da^2 = a$. Then

(1) $ab = ba^2b = da,$

So that

(2) $ab^2 = dab = d^2a.$

From (1) we have also

(3) $aba = da^2 = a = a^2b = ada.$

Now put $c = ab^2$. It follows then from (1), (2), (3), that

$ac = adab = ab = da\ daba = ca, a^2c = aca = aba = a,$

$ac^2 = dac = dab = c,$ as desired .

Suppose next c' be any element which satisfies the same equalities as c: $ac' = c'a, a^2c' = a,\ a^2c' = c'$. Then, be replacing b, d in (2) by c, c' respectively , we get $c = ac^2 = c'^2a = c'$, showing the uniqueness of c.

For the proof of the last assertion, let z be any element s.t. $az = za$. Then we have first $caz = cza = cza^2c = ca^2zc = azc = zac$, i.e., z commutes with $ca = ac$. It follows from this now $cz = c^2az = czca = cazc = zcac = zc,$ and this completes the proof. ■

corollary 2.19[2] :

Let a be a st. π −reg. element of A. Suppose that a^n is right reg.. Then a^n is in fact st. reg., and moreover there exists an element c s.t. $ac = ca$ and $a^{n+1}c = a^n$.

Corollary 2.20 [3]:

L et $\mathcal{R}$ be a st. π-reg. rg. and s∈ $\mathcal{R}$. Then ∃ $n \geq 1$and $a \in \mathcal{R}$ s.t. $s^n = s^{2n}a,\ sa = as$ and $a^2s^n = a$.

Theorem 2.21 :

Let $\mathcal{R}$ be a rg. and $\{S_i\}_{i\in I}$ a collection of st. π-reg. subrg.s. Then $\bigcap_{i\in I} S_i$ is st. π-reg..

Proof: Let $\in S$. U sing one of the S_i we can find $n \geq 1$ and $a \in S_j$ s.t. $s^n = s^{2n}a, sa = as\ and\ a^2s^n = a$. Now consider S_i For some $m \geq 1$ and $b \in S_j$ there is a solution for $s^{nm} = s^{2mn}b,\ s^{nm}b = bs^{nm}, b^2s^{nm} = b$. Further $s^{nm} = s^{2nm}a^m, s^{nm}a = as^{nm}\ and\ a^{2m}s^{nm} = a^m$.

By corollary 2.26, $b = a^m \in S_j$. From $a = a^2s^n$ it follows that $a = a^m s^{(m-1)n} \in S_j$ if $m \geq 1$. If $m = 1$, $b = -a$ already. In any case $a \in S_j$. ■

Lemma 2.22 [2]:

Let a be a st. π-reg. element of index n, and c an element s.t. $ac = ca$ and $a^{n+1}c = a^n$(as in corollary 2.19. Then $a - a^2c$ is a nilpotent element of index n.

We now obtain from corollary (2.19) and lemma (2.22), immediately the following.

Theorem 2.23:

Let $\mathcal{R}$ be a ring and let $a \in \mathcal{R}$ be a st. π-reg. element. Then there exists elements $u \in \mathcal{R}$ and $h \in \mathcal{R}$ s.t.

1. u is invertible. 2. $uh = hu = a$ 3. h is w-idempotent.

Proof: ← By corollary (2.19), $\exists\, c \in \mathcal{R}$ and $n \in \mathcal{R}$ s.t. $a^{n+1}c = a^n$ and $ca = ac$. Then we have

$a^n = a^{n+1}c = a^{n+2}a^2 = \cdots = a^{2n}c^n = a^nc^na^n$.

Let $w = a^nc^n$.

Then $w^2 = w$ and the elements a , c and w commute with ∀ other.

We also have $acw = ac(a^nc^n) = (a^{n+1}c)c^n = a^nc^n = w$

and $a^nw = a^nc^na^n = a^n$.

Let $u = aw + (1 - w)$

and $h = w + a(1 - w)$ then $uh = hu$.

And $uh = [aw + (1 - w)][w + a(1 - w)] = aw^2 + a(1 - w)^2 = aw + a - aw = a$.

Also $h^n = [w + a(1 - w)]^n = w^n + a^n(1 - w)^n$

$= w + a^n(1 - w) = w + a^n - a^nw = w$.

Thus g is an w-idempotent.

Finally ; let $z = [cw + (1 - w)]$ then $zu = uz$ and

$uz = [aw + (1 - w)][cw + (1 - w)] = acw^2 + (1 - w)^2 = w + (1 - w) = 1$.

Therefore u is invertible. ■

Corollay 2.24 :

Let $\mathcal{R}$ be a st. π-reg. ring and let $a \in \mathcal{R}$, then ∃elements $u \in \mathcal{R}$ and $h \in \mathcal{R}$ s.t. 1. u is invertible. 2. $uh = hu = x$. 3. h is an w-idempotent.

Moreover, if A is a rg. s.t. for every element $a \in A$ ∃elements $u \in A$ and $h \in A$ satisfying conditions (1),(2) and (3), then A is st. π-reg..

Proof: ←The first assertion directly from theorem (2.23).

The second assertion s.t. let $a \in A$ and there exists elements $u \in A$ and $h \in A$ satisfying condition (1) , (2) and (3) for integer $n > 0$ s.t.

$h^{2n} = h^n$. Then $a^n = u^nh^n = u^{2n}u^{-n}h^{2n}$

$= a^{2n}u^{-n} = a^{n+1}(a^{n-1}u^{-n})$

And thus S is st. π −reg..

Remark 2.25:

We list have other useful relations of the elements use in the proof of theorem (2.23).

Let a is st. π-reg. elements and a rg. $\mathcal{R}$, and let $n \in N$ and a, c and w in $\mathcal{R}$ be the same in the proof of Theorem 2.23.

Thus we have $a^{n+1}c = a^n$, $ac = ca$

$w = a^nc^n$

$acw = w$,

$a^nw = w$

and a, c and w commute with ∀ other

set $u = aw + (1 - w)$

$v = aw - (1 - w)$

Then u and v and invertible with inverse

$u^{-1} = cw + (1 - w)$

$v^{-1} = cw - (1 - w)$

Finally , $a(1 - w)$ is nilpotent with $(a(1 - w))^n = 0$

It is st. π-reg.. ■

It is clearly consequence of corollary (2.24), is another proof of the result that $J(R)$, the Jacobson radical of R is nil when R is st. π-reg.. Since 0 is the only idempotent in J(R), nilpotent elements only w-idempotent in J(R).

← If $a \in J(\mathcal{R})$ and h is an w-idempotent in the decomposition of a, then h is also in $J(\mathcal{R})$. Hence h (and hence a) is nilpotent.

In the following we will present the very important theorem.

Theorem 2.26 :

Let $\mathcal{R}$ be a st. π-reg. rg. if 2 is a unit in $\mathcal{R}$, then for all element of $\mathcal{R}$ can be expressed as a sum of two units.

Proof: Suppose $a \in \mathcal{R}$. Then as in the proof of Theorem 2.23, $\exists c \in \mathcal{R}$ and $n \in N$ s.t. $ac = ca$ and $a^{n+1}c = a^n$.

Let elements $w, u, v, u^{-1}$and v^{-1} in $\mathcal{R}$ be define as in remarks following colloary 2.24, Since v commutes with

$a(1 - w)$, we have that $2^{-1}v + a(1 - w)$ is a unit.

Thus $2^{-1}u + [2^{-1}v + a(1 - w)] =$

$2^{-1}\big(aw + (1 - w)\big) + 2^{-1}[aw - (1 - w)] +$

$a(1 - w) = aw + a(1 - w) = a$.

Hence a is the sum of two units. ■

Now, Let $\mathcal{R}$ be a rg., and let $U(\mathcal{R})$ denoted the subrg. of $\mathcal{R}$ generated by the units of $\mathcal{R}$.

Thus , Theorem 2.24, shows that if $\mathcal{R}$ is st. π-reg.. And 2 is a unit of $\mathcal{R}$, then $U(\mathcal{R}) = \mathcal{R}$.

Proposition 2.27:

Let $\mathcal{R}$ be a st. π-reg. rg. and let A be a subrg. of $\mathcal{R}$.

If $U(\mathcal{R}) \leq A$, then A is st. π-reg..

Proof: ←Let $a \in A$. Thus $a = uh$, where $u \in \mathcal{R}$, $h \in \mathcal{R}$, u is invertible, $uh = hu$, and h^n is idempotent for some integer $n > 0$. Then $u \in A$, $u^{-1} \in A$ and so $h = u^{-1}a \in A$. Thus the factorization in $\mathcal{R}$is also a factorization in A, and so by corollary (2.24), A is st. π-reg.. ■

Proposition 2.28:

Every element a in a st. π-reg. rg. $\mathcal{R}$ is unit st. π-reg.; i.e. a has a generalized invers of st. π-reg. which is invertible.

Proof: Let $a \in \mathcal{R}$. Then as in the proof of theorem (2.23), $\exists\, c \in \mathcal{R}$ and $n \in N$ s.t. $ac = ca$ and $a^{n+1} = a^n$ let elements w and u in $\mathcal{R}$ be defined an in the remarks following corollary (2.24).

Then $a^{n+1}u^{-1} = a^{n+1}[cw + (1 - w)]$

$= a^nacw + a^{n+1} - aa^nw$

$= a^nw + a^{n+1} - a^{n+1}$

$= a^n$. ■

Definition 2.29 [5]:

A ring R is said to be reduced if it has no nonzero nilpotent elements.

Proposition 2.30:

Let $\mathcal{R}$ be a reduced st. π-reg. rg.. Then $\mathcal{R}$ is a reg. rg..

Proof : Let $a \in \mathcal{R}$. Then as in the proof of Theorem 2.23, $\exists\, c \in \mathcal{R}$ and $n \in \mathcal{R}$ s.t. $ac = ca$ and $a^{n+1} = a^n$.

Let $w = a^nc^n$, then as in the remark following corollary 2.24 ,

$a(1 - w)$ is nilpotent.

Hence $a = a^{n+1}c^n = a(a^{n-1}c^n)a$. ■

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بعض النتائج حول الحلقات المنتظمة القوية من نمط - π

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الملخص

في هذا البحث قمنا بدراسة الحلقة المنتظمة القوية من نمط –π وبعض الخصائص التي تعطي ايضا بعض النتائج الجديدة حول الحلقة المنتظمة القوية من نمط –π وارتباطه بحلقات اخرى .