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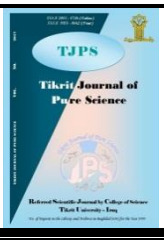

The Necessary Condition For Fixed Points In The Inverse Limits Spaces

Ali H. Ali[1] and Faruq A. Mena[2]
[1,2] Mathematics Department, Faculty of Science, Soran University, Soran, Erbil, Kurdistan Region, Iraq,

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Corresponding Author*:
Ali H. Ali
alihaligrtky@gmail.com

ABSTRACT
A point p in the inverse limit space is said to be a cut point of this space when excluded from it, when the number of the components of that space increases. Therefore, this study aims at finding the necessary condition for fixed points in the inverse limit space to be cut points. Then, for applying the main theorem with some conditions, a sequence of upper semi continuous can be employed as a bonding function to get a union of continua as a generalized inverse limit space if there is a generalized inverse limit for each of them separately.

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الشرط الضروري للنقاط الثابتة في فضاءات الغايات المعكوسة

علي حسن علي[1]، فاروق عبدالله مينه [2]

[2،1] قسم الرياضيات، كلية العلوم، جامعة سوران، سوران، اربيل، العراق.

الملخص

يقال عن النقطة p في فضاء الغاية المعكوسة بأنها نقطة قطع لفضاء غاية المعكوسة اذا استبعدناها من ذلك الفضاء عندما يزداد عدد قطع الفضاء. كذلك، تم ايجاد الشرط الضروري لتكون النقاط الثابتة في فضاء الغاية المعكوسة نقاط قطع. بالاضافة إلى ذلك وكتطبيق للمبرهنة الرئيسة مع بعض الشروط فإننا نستطيع ايجاد متتابعة من الحدود العليا شبه المستمرة مثل دالة الترابط للحصول على على اتحاد الاستمرارية كتعميم لفضاء الغاية المعكوسة إذا كنّا قد عممنا الغاية المعكوسة لكل منها بصورة منفصلة.

الكلمات المفتاحية: معكوسة الغايات، كونتينيم، النقاط الثابتة، دوال متعددة القيم، دوال شبه متصل علوي.

1. Introduction

In 2004, Mahavier introduced a generalized inverse limit on intervals [1]. Later, in 2006, Ingram and Mahavier introduced this limit on the compact sets [2]. Recently, researchers published a number of results about some continuum properties in an inverse limit space. In 2015, Banic and Martinez found the universal dendrite D_3 as the generalized inverse limit space [3]. In 2022, Corona studied dendrites as the generalized inverse limit space [4], while Marsh studied atriodic tree like continua as inverse limits on [0,1] [5]. Currently, the generalized inverse limit is a powerful tool in the study of continuum theory.

A topological space X is said to be continuum if it is a nonempty, connected, compact and metric space. A subcontinuum is a subset of the continuum. In this regard, 2^X denotes the hyperspace of X when X is a continuum. A set valued function $f: X \to 2^Y$ is said to be an upper semi-continuous function if for each element x in the space X and all open subsets V in the space Y, which contains f(x), there is an open set U in X which contains x such that for each element t in U, then $f(t) \subseteq V$. If X and Y are compact metric spaces and $f: X \to 2^Y$ is a set valued function, then f is an upper semi-continuous function if and only if its graph $G(f) = \{(x, y): y \in f(x)\}$ is a closed subset in $X \times Y$ [6, p. 3]. Let X and Y be compact Hausdorff metric spaces and $f: X \to Y$ be a continuous function. The function f is said to be monotone if for each $y \in Y$ the inverse image of y ($f^{-1}(y)$) is a continuum. Let $(X_i)_{i\in N}$ be a sequence of continua and $f_i: X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function. The generalized inverse limit space of a sequence $\{X_i, f_i\}$ is denoted by $\varprojlim\{X_i, f_i\}$ and defined by $\varprojlim\{X_i, f_i\} = \{(x_i)_{i=1}^{\infty}: x_i \in f_i(x_{i+1}) \text{for all } i \in N\}$. All inverse limits in this study are generalized inverse limit spaces. The distance between elements x and y in the inverse limit space is defined by $d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$, when $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are elements in $\varprojlim\{X_i, f_i\}$ and d_i is a metric space on X_i bounded by 1. More information about inverse limits of continua having set valued upper semi-continuous bonding functions defined on [0,1] can be founded in [7] and [6].

2. Important Definitions and Examples in Continuum Theory

This section presents the important definitions. Most of these definitions are found in Macias (2005) [8], Nadler (1978) [9] and Nadler (1992) [10].

Definition 2.1: The Gehman Dendrite of order n is denoted by G_n, defined by a dendrite such that all of its ramification points are of order n and the set of end points $E(G_n)$ is homeomorphic to the Cantor ternary set.

Definition 2.2: Harmonic fan continuum is a continuum defined by a union of arcs, joining the point (0,1) to

$(\frac{1}{n}, 0)$, $n \in N$ together with the arc $A = \{(0, y), 0 \leq y \leq 1\}$. It is not a locally connected continuum because all points in the limit bar are non-locally connected points. It is not a dendrite because it is not locally connected.

Example 2.3: The continuum F_ω is a dendrite defined by the union of sequence of straight lines $\{l_n\}_{n+1}^{\infty}$ such that $\cup_{n=1}^{\infty}\{l_n\}$ has only one ramification point of order ω and $\lim_{n\to\infty} H_d(l_n, 0) = 0$, [7] and [8].

Definition 2.4: Let $m \in \{3,4,\ldots,\omega\}$, the universal dendrite of order m and be denoted by D_m such that all of its ramification points are of order m and for each arc subset $A \subset D_m$, the set of ramification points in the dendrite D_m located in A is dense in A.

Definition 2.5: The Hilbert cube is a continuum which is homeomorphic to the product $Q = \prod_{i=1}^{\infty} I_i$ when I_i is the united closed interval [0,1].

Definition 2.6: A mapping $f: X \to Y$ is said to be an ϵ map if for each element y in Y, the diameter $\dim(f^{-1}(y))$ is less than ϵ.

Definition 2.7: Let X be a continuum and P be a topological property. X is said to be P like if there exists an ϵ map from X to a continuum having the property P.

Definition 2.8: The topologist's sine curve (The $\sin(\frac{1}{x})$ continuum) is a continuum which is homeomorphic to the closure of $\{(x, y) \in R: x \in (0,1], y = \sin(\frac{1}{x})\}$ as shown in Figure 1.

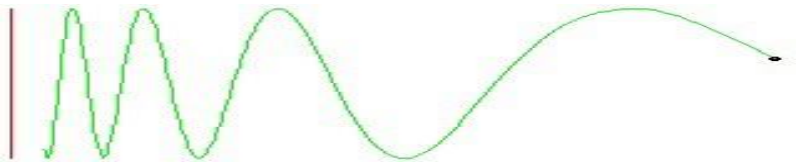

Figure 1: Topologist's Sine Curve Inverse Limits

The topologists sine curve is an arc like continuum because for all $\epsilon > 0$, an ϵ map can be found from topologist's sine curve to an arc [see: Macias 2005, p106, Example 2.4.5]. It is homeomorphic to the inverse limit of a single bonding mapping over unit interval factor spaces such that $f(x) = 2x$ when $0 \leq x \leq \frac{1}{2}$; $f(x) = \frac{3}{2} - x$ when $\frac{1}{2} < x \leq 1$. It is an arc like continuum with two arc components. More details about that inverse limit are found in [7, p11, Example 16]. It is irreducible between $(1, \sin(1))$ and $(0, y)$, $-1 < y \leq 1$. A confluent image of the topologist's sine curve is an arc or a continuum which is homeomorphic to the topologist's sine curve.

Definition 2.9: The double topologist's sine curve with one limit bar is defined by $\{(x, y) | y = \sin(1/x), -1 \leq x \leq 1, x \neq 0\} \cup \{(0, y) | -1 \leq y \leq 1\}$. It is an arc like continuum with three arc components and is irreducible between the points $(-1, \sin(-1))$ and $(1, \sin(1))$ [see: Macias (2005), Example 2.4.6].

Definition 2.10: The double topologist's sine curve with two limit bars is defined by $\{(x, y) | y = \sin(\frac{1}{1-|x|}), -1 < x < 1\} \cup \{(-1, y) | -1 \leq y \leq 1\} \cup \{(1, y) | -1 \leq y \leq 1\}$. It is an arc like continuum with three arc components and is irreducible between the points $(-1, y_1)$ and $(1, y_2)$, $-1 \leq y_1, y_2 \leq 1$, [Macias (2005), Example 2.4.6].

Definition 2.11: The Warsaw circle continuum is a union of a continuum X and a continuum Y where X is the topologist's sine curve and Y is an arc joining points $(1, \sin(1))$ and $(0, -1)$. It is a circle like continuum [8]. A double Warsaw circle is a union of a double topologist's sine curve continuum and an arc as shown in Figure 2.

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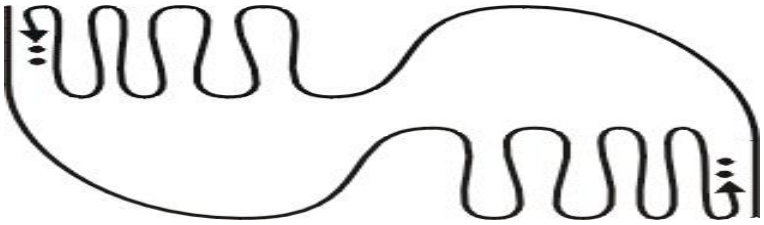
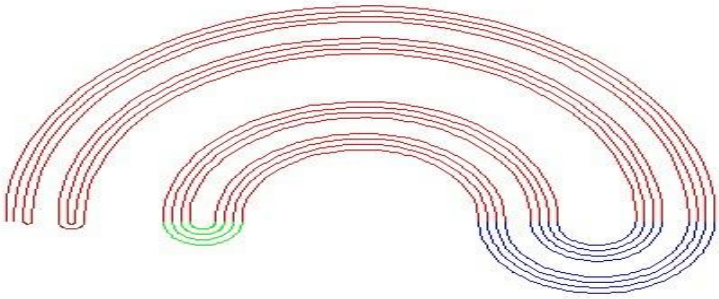


Figure 2 Warsaw circle with two limit bars

Definition 2.12: The Knaster, BJK or buckethandle continuum is denoted by K as shown in Figure 3 and defined by the following: the non-negative ordinal set of all semi circles with $(\frac{1}{2}, 0)$ center and intersect Cantor set C; the non-positive ordinal set of all semi circles such that $\forall n \in N$, with center $(\frac{5}{2.3^n}, 0)$ and intersect all Cantor set points in the interval $[\frac{2}{3^n}, \frac{1}{3^{n-1}}]$ [11, p204-205].

Figure 3: Knaster continuum



The name BJK continuum came from the first letters of these three famous mathematicians: Brouwer, Janiszewski and Knaster. They constructed such continuum in different ways. It is known that the Knaster continuum is an indecomposable continuum. It is an arc like continuum. If K is a Knaster continuum and K′ is the reflection of K around the origin, $K \cup K'$ is a continuum and arc like but it is not indecomposable continuum. Knaster or BJK continuum can be considered as the inverse limit of bonding mapping defined on factor spaces unit open interval I, s.t $f(x) = 1 - |2x - 1|$, $x \in I$ [7, p15, Example 22]. A double Knaster (Buckethandle) continuum as shown in Figure 4 is defined as the inverse limit with bonding mapping f_n defined on intervals such that $f(x) = 3x$ when $0 \leq x \leq \frac{1}{3}$; $f(x) = 2 - 3x$ when $\frac{1}{3} < x \leq \frac{2}{3}$; $f(x) = 3x - 2$ when $\frac{2}{3} < x \leq 1$. It is indecomposable arc like continuum as well.

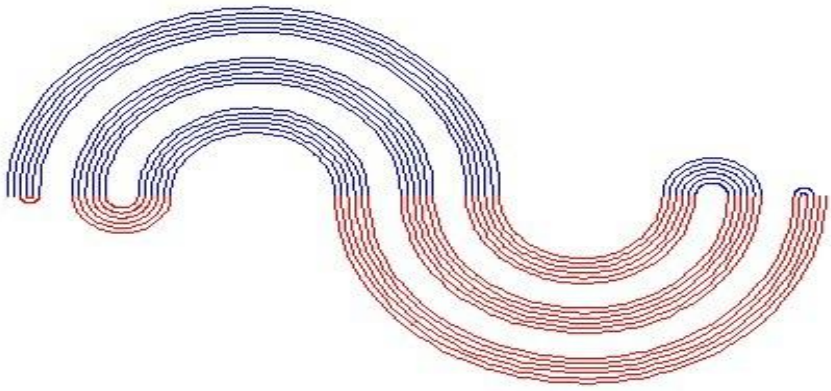


Figure 4: A Double Knaster continuum

Definition 2.13: The Menger continua represent a universal continuum M_n^m, $1 \leq m \leq n$, and defined as follows:

Let $L_o = I^n$. It is defined inductively. Let L_k be a collection of cubes and defined for all $k \geq 0$. Divide cubes D in L_k into $3^{n(k+1)}$ congruent cubes such that the length edge of the new cubes be $\frac{1}{3^{n(k+1)}}$. If $F_{k+1}(D)$ is the collection of cubes intersect faces of n dimensional D, then $F_{k+1} = \cup\{F_{k+1}(D): D \in F_k\}$. Define M_m^n by $M_m^n = \cap_{i=0}^{\infty}(\cup F_i)$. Sierpinski universal plane curve is M_1^2.

Let $X_o = I^2$. Divide X_o into nine congruent squares and remove the middle one to get $X_1 = X_o \backslash \text{int}([\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}])$. Similarly, for the rest of the remaining eight squares to get X_2. This process is continued in this way to get $X_3, X_4, \ldots$. The intersection $X = \cap_{i=1}^{\infty}\{X_i\}$ of all X_i, $i = 1,2, \ldots$ is said to be Sierpinski Universal Curve. It is a locally connected continuum curve and it does not have any cut points. It is embedded in I^2 so that $R\backslash X$ has these components $K_1, K_2, \ldots$ such that $\text{diam}(K_i) \to 0$, $b(K_i) \cap b(K_j) = \emptyset$ for $i \neq j$ and the boundary of K_j is a simple closed curve and the union of $\cup_{i=1}^{\infty} K_i$ is dense in X^2.

Definition 2.14: A. topological space X is said to be a locally connected continuum if for each element p in X and each neighborhood U of p, there exists a continuum neighborhood of p in U [11].

Definition 2.15: A dendrite X is said to be a locally connected continuum if it does not have any simple closed curve. Dendrites are hereditary unicoherent that is the intersection of any of its two sub continua is a continuum.

Definition 2.16: Let $f: X \to X$ be a continuous function. A point p in X is said to be a fixed point if $p = (a, a)$ is an element in the graph $G(f)$ in X^2 [12].

Definition 2.17: Let $f: I \to 2^I$ be an upper semi continuous function and $X = \varprojlim\{I, f\}$ be the generalized inverse limit space. A point p in X is said to be a fixed point if $p = (a, a, a, a, a, \ldots)$ where $a \in I$ and $a = f^{-1}(a)$.

3. Main Theorems

This section clarifies how the fixed points in the inverse limit space are considered cut points under some restrictions. It starts with some basic definitions. A point p in a dendrite D is said to be an endpoint of the dendrite D if for any two arcs containing p there is another point in the intersection of them. The point p in the dendrite D is an ordinary point of D if $D \cap \{p\}^c$ has only two components, and the point p is said to be a ramification point of the dendrite D if $D \cap \{p\}^c$ has n components for $n \geq 3$. The order of a point p in a dendrite D is n, where n is an element in the set $N \cup \{\omega\}$, if $D \cap \{p\}^c$ has n components. These notations are used: E(D) is used for the set of end points of the dendrite D and R(D) is used for the set of ramification point of D. The dendrite G_n or Gehman dendrite or order n is the dendrite where all of its ramification points are of order n and its E(D) is homeomorphic to the Cantor set [13, Theorem 4.1].

The first main theorem in this study is as follows:

Theorem 3.1 Let $f: [0,1] \to 2^{[0,1]}$ be an upper semi continuous function such that $G(f) = \cup_{i=1}^{n}[G(f_i)]$ is a continuum, where $f_i|_{[a_{i-1}, a_i]}: J_i = [a_{i-1}, a_i] \to 2^{J_i}$, $i = 1, \ldots, n-1$ is the restriction of f on J_i, $G(f_i) \cap y_i = (a_i, a_i)$ and $G(f_i) \cap x_i = (a_i, a_i)$ where $y_i(x) = a_i$ and $x_i(y) = a_i$ are horizontal and vertical line segments, respectively. If $y_i(x) \cap G(f_i)$ is a non-degenerate, then $y_i(x) \cap G(f_k)$ is degenerate for $k \neq i$. If the inverse limit is a continuum and points $p_i = (a_i, a_i, a_i, \ldots)$ $i = 1, 2, \ldots, n-1$ are locally connected points in $\varprojlim\{I, f\}$, then they are cut points of $\varprojlim\{I, f\}$ and $\varprojlim\{I, f\} = \cup_{i=1}^{n} \varprojlim\{J_i, f_i\}$.

Proof. Since for each $t \in \text{int}(J_i)$, $f^{-1}(t) \in \text{int}(J_i)$, so for each $t \in f_i(J_i) \cap \text{int}(J_i)$, $f^{-1}(t) \in f_i(J_i) \cap \text{int}(J_i)$. So, $G(f_i)$ and $G(f_i^{-1})$ are subsets of I_i^2. Let $(x, y) \in \text{int}(G(f_i))$. It is clear from the definition of f_i that $a_{i-1} < x, y < a_i$. Let $\epsilon_1 = \min\{\frac{|x-a_{i-1}|}{2}, \frac{|a_i-x|}{2}\}$ and $\epsilon_2 = \min\{\frac{|y-a_{i-1}|}{2}, \frac{|a_i-y|}{2}\}$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. It is easy to see that the open ball

$B_\epsilon(x,y)$ is a proper subset of $\text{int}(I^2)$, so it does not contain any point in J^2_{i-1} nor J^2_{i+1} or any point in J^2_k where $k \neq i$. It follows that (x,y) is not a limit point of any point J^2_k where $k \neq i$. Consequently, any point $G(f_i)$ does not belong to the derive set of J^2_k where $k \neq i$. Thus, $G(f_i)$ does not contain any point of $G(f_k)$ for $k \neq i$. Therefore, the intersection of $G(f_i)$ and the closure of $G(f_k)$, $\overline{((G(f_k)}$ is empty for all $i \neq k$. Let $p_i = (a_i, a_i, \dots)$ be a point in $\varprojlim\{I, f_i\}$ such that $i \in \{1,2,3,\dots, n-1\}$. Note that $\pi_{m,m+1}(p_i) = (a_i, a_i)$, for $i \in N$. To prove that p_i is a cut point of the inverse limit space, it is necessary to prove that there exists an open neighborhood $B_\epsilon(p_i)$ in I^2 of p_i such that $(B_\epsilon(p_i) \cap \varprojlim\{I, f_i\})\backslash\{p_i\}$ is disconnected. Let $\epsilon_1 = \min\{\frac{|x-a_{i-1}|}{2}, \frac{|a_i-x|}{2}\}$ and $\epsilon_2 = \min\{\frac{|y-a_{i-1}|}{2}, \frac{|a_i-y|}{2}\}$. Let $U \cap \varprojlim\{I, f_i\}$ be a neighborhood of p_i in the inverse limit space where $U = \Pi_{i=1}^{m-1} I \times (p_k - \epsilon_1, p_k + \epsilon_1) \times (p_k - \epsilon_2, p_k + \epsilon_2) \times Q$, where $Q = \Pi_{i=m+2}^{\infty} I$. It is clear that $(U_1 \cap \varprojlim\{I, f_i\})$ and $(U_2 \cap \varprojlim\{I, f_i\})$ are disjoint at p_i where $U_1 = \Pi_{i=1}^{m-1} I \times (p_k, p_k + \epsilon_1) \times (p_k, p_k + \epsilon_2) \times Q$, $U_2 = \Pi_{i=1}^{m-1} I \times (p_k - \epsilon, p_k) \times (p_k - \epsilon, p_k) \times Q$, where $Q = \Pi_{i=m+2}^{\infty} I$ and their union is U. Using [13, Theorem 26.5, p. 192], it is obtained that U_1 and U_2 are separated. Since the image and the pre image of any point in J_i will stay in J_i, so $\varprojlim\{J_i, f_i\}$ is homeomorphic to $\varprojlim\{J_i, f_i|_{[a_{i-1}, a_i]}\}$. It is obtained that $\varprojlim\{I, f\}$ is homeomorphic to $\cup_{i=1}^{n} \varprojlim\{J_i, f_i|_{[a_{i-1}, a_i]}\}$. This represents the end of the proof.

4. Applications

This section presents several applications of Theorem 3.1. It can be proved that the union of finitely many inverse limit continua is the inverse limit of a single bonding map on $[0,1]$ under some restrictions.

Example 4.1 Let a set valued function $f: [0,1] \to 2^{[0,1]}$ be an upper semi continuous function defined by:

$$f(x) = \begin{cases} \{0, \frac{1}{2}\} & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1}{4} & \text{if } x = \frac{1}{4} \\ \{0, \frac{1}{4}\} & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ 2x - 1 & \text{if } x \in \{\frac{1}{2}, \frac{3}{4}\} \\ -x + \frac{7}{4} & \text{if } x \in \{\frac{3}{4}, 1\} \end{cases}$$

The point $(\frac{1}{2}, \frac{1}{2})$ is the separated point of the inverse limit space and the inverse limit space $\varprojlim\{I, f\}$ is the union of $\varprojlim\{I_1, f_1\}$ and $\varprojlim\{I_2, f_2\}$ where $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$.

Proof: Note that the current bonding upper semi continuous function satisfies the requirement of Theorem 3.1. Again the inverse limit space is the union of $\varprojlim\{I_1, f_1\}$ and $\varprojlim\{I_2, f_2\}$ where $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. Since the graph of bonding upper semi continuous function of f_1 and that found in [6, Example 2.22] are Markove like in the same pattern, so they have a homeomorphic inverse limit space, which is G_3 [15]. In the same way, $\varprojlim\{I_2, f_2\}$ and the inverse limit in [7, Example 16] are homeomorphic, representing the closure of a topological array R and $\underline{R}\backslash R$ as shown in Figure 1. The inverse limit space is homeomorphic to the union of the above inverse limits by identifying the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ as shown in Figure 5. The point $(\frac{1}{2}, \frac{1}{2}, \dots)$ is a separated point of the inverse limit space and the inverse limit space $\varprojlim\{I, f\}$ is the union of $\varprojlim\{I_1, f_1\}$ and $\varprojlim\{I_2, f_2\}$ where $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$.

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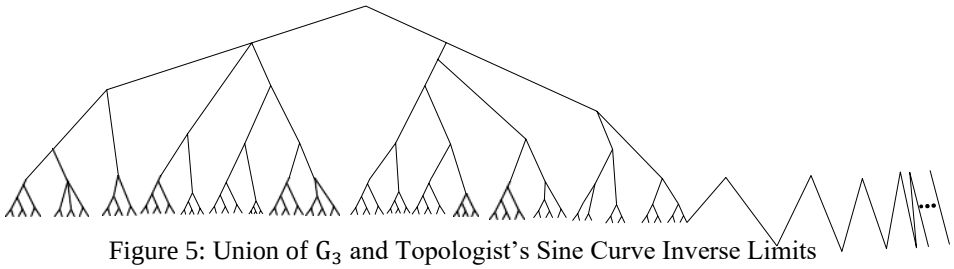

Figure 5: Union of G_3 and Topologist's Sine Curve Inverse Limits

Example 4.2 Let f_1 be defined as in [6, Example 2.17, p36]; f_3 is defined as in [7, Example 16, p11]; f_2 is defined as in equation 4.1: Let a set valued function f: $[0,1] \to 2^{[0,1]}$ be an upper semi continuous function as in equation 4.2.

$$f_2(x) = \begin{cases} 3x & \text{if} \quad x \in [0, \frac{1}{3}] \\ -3x + 2 & \text{if} \quad x \in [\frac{1}{3}, \frac{2}{3}] \\ 3x - 2 & \text{if} \quad x \in [\frac{2}{3}, 1] \end{cases} \tag{4.1}$$

$$f(x) = \begin{cases} \frac{f_1(3x)}{3} & \text{if} \quad x \in [0, \frac{1}{3}] \\ \frac{1}{3} + \frac{f_2(3x-1)}{3} & \text{if} \quad x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{2}{3} + \frac{f_3(3x-2)}{3} & \text{if} \quad x \in [\frac{2}{3}, 1] \end{cases} \tag{4.2}$$

Using Theorem 3.1, the points $(\frac{1}{3}, \frac{1}{3}, \dots)$ and $(\frac{2}{3}, \frac{2}{3}, \dots)$ are separated points of the inverse limit space and the inverse limit space is homeomorphic to $\varprojlim\{I_1, f_1\} \cup \varprojlim\{I_2, f_2\} \cup \varprojlim\{I_3, f_3\}$ as shown in Figure 6.

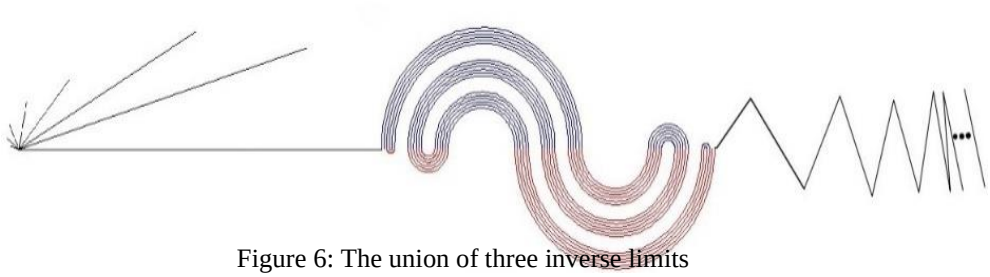

Figure 6: The union of three inverse limits

5. Conclusion

In conclusion, this study found the necessary condition for some points in the set of fixed points in the generalized inverse limit space to be cut points. As for the application of the new main theorem, points in the inverse limit space can be easily defined as cut points from their graph of upper semi continuous bonding functions. In addition, a sequence of upper semi continuous bonding functions on $[0,1]$ can be easily invented to obtain a union of two or more than two continua by knowing the inverse limit of each one of them separately.

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