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Weakly Approximately primary submodules and Related Concepts

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ABSTRACT

Let R be a commutative ring with identity, and $\mathcal{H}$ be a unital left R-module. In this paper the concept of weakly approximately primary submodule are introduced as a new generalization of a weakly primary submodule, also it is a generalization of weakly prime submodule. Various basic properties of weakly approximately primary submodules are studied. Moreover, many characterizations and examples of this concept are investigated.

1. Introduction

Throughout this research all rings are commutative with identity and all modules are unitary. Weakly primary submodule was first introduced in 2005 by Atani and Farzalipour, where, 'a proper sub module E of an R-module $\mathcal{H}$ is a weakly primary submodule if wherever $0 \neq rh \in E \; for \; r \in R, h \in \mathcal{H}$, implies that $h \in E \; or \; r^n\mathcal{H} \subseteq E$ for some positive integer n' [1]. And, 'aproper submodule E of an R-module $\mathcal{H}$ is called a weakly prime if wherever$0 \neq rh \in E \; for \; r \in R, h \in \mathcal{H}$, implies that $h \in E \; or \; r\mathcal{H} \subseteq E$ ' [2]. 'It is well known that every weakly prime submodule is a weakly primary' [1]. 'Many auther studied weakly primary submodules see for examples' [3.4]. In this note we generalized the concept of weakly primary sub module to the concept weakly approximately primary submodule, where a proper submodule E of an R-module $\mathcal{H}$ is a weakly approximately primary if wherever$0 \neq rh \in E \; for \; r \in R, h \in \mathcal{H}$, implies that $h \in E + Soc(\mathcal{H}) or \; r^n\mathcal{H} \subseteq E + Soc(\mathcal{H})$ for some positive integer n. In this part of the paper we recall some basic definitions, to be very important in the sequal. 'The socal of a module $\mathcal{H}$ denoted by $Soc(\mathcal{H})$ is the intersection of all essential submodules of $\mathcal{H}$' [5]. 'Where a nonzero submodule N of $\mathcal{H}$ is an essential if $N \cap L \neq 0$ for all nonzero submodule L of $\mathcal{H}$' [6]. 'An R-module $\mathcal{H}$ is called torsion free if $T(\mathcal{H}) = \{h \in \mathcal{H}: ah = 0 \; for \; some \; 0 \neq a \in R\} = (0)$ and $\mathcal{H}$ is called torsion if $T(\mathcal{H}) = \mathcal{H}$' [7]. 'Let E be a submodule of $\mathcal{H}$ and J is an ideal of R define a submodule $[E :_R J] = \{h \in \mathcal{H}: hJ \subseteq E\}$ with $E \subseteq [E :_R J]$and $[E :_w R] = E$, $[I :_w R] = I$' [8]. 'A zero divisor on an R-module $\mathcal{H}$ is an element $a \in R$ for which there exists a nonzero element $h \in \mathcal{H}$ such that $ah = 0$' [9].

2. Properties of Weakly Approximaitly primary submodules

This section devoted to introduce the definition of weakly approximately primary submodule and illustrate some properties, examples and characterizations of it.

Definition (2.1)

A proper sub module E of an R-module $\mathcal{H}$ is a weakly approximately primary (Brevily wapp-primary) submodule of$\mathcal{H}$, if $0 \neq ah \in E, where \; a \in R, h \in \mathcal{H}$, implies that $h \in E + Soc(\mathcal{H}) \; or \; a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, that is $a^n \mathcal{H} \subseteq E + Soc(\mathcal{H})$ for some positive integer n. And an ideal J of a ring R is a weakly approximately primary ideal if J is a weakly approximately primary R-submodule of an R-module R.

Remark (2.2)

Every weakly primary submodule of an R-module $\mathcal{H}$ is a wapp-primary submodule of $\mathcal{H}$ but the converse is not true in general.

Proof

Let E be a weakly primary submodule of $\mathcal{H}$, and $0 \neq ah \in E, where\ a \in R, h \in \mathcal{H}$. Since E is a weakly primary submodule of $\mathcal{H}$, then we have $h \in E \subseteq E + Soc(\mathcal{H})$ or $a^n \mathcal{H} \subseteq E \subseteq E + Soc(\mathcal{H})$. Thus E is a wapp-primary submodule of $\mathcal{H}$.■

Consider the following example for the converse

Example (2.3)

Let $\mathcal{H}$ be the Z-module Z_{60}, and $E = <\bar{6}> = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}, \overline{36}, \overline{42}, \overline{48}, \overline{54}\}$ be the submodule of $\mathcal{H}$. $Soc(Z_{60}) = <\bar{2}> \cap <\bar{1}> = <\bar{2}>$ (since the only essential submodule of Z_{60} are the submodule $<\bar{2}>$ and $Z_{60} = <\bar{1}>$). E is not weakly primary submodule of Z_{60}, since $0 \neq 3 \cdot \bar{2} \in <\bar{6}>$, $3 \in Z, \bar{2} \in Z_{60}$, but $\bar{2} \notin E = <\bar{6}>$ and $3 \notin \sqrt{[<\bar{6}> :_z Z_{60}]} = \sqrt{6Z} = 6Z$

Furthermore E is wapp-primary submodule of Z_{60}, since $0 \neq 3 \cdot \bar{2} \in E = <\bar{6}>$, for $3 \in Z$, $\bar{2} \in Z_{60}$, implies that $\bar{2} \in E + Soc(Z_{60}) = <\bar{6}> + <\bar{2}> = <\bar{2}> = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}, \overline{24}, \overline{26}, \overline{28}, \overline{30}, \overline{32}, \overline{34}, \overline{36}, \overline{38}, \overline{40}, \overline{42}, \overline{44}, \overline{46}, \overline{50}, \overline{52}, \overline{54}, \overline{56}, \overline{58}\}$

That is wherever $0 \neq ah \in E = <\bar{6}>$, where $a \in Z, \bar{h} \in Z_{60}$, implies that $\bar{h} \in <\bar{6}> + Soc(Z_{60}) = <\bar{6}> + <\bar{2}> = <\bar{2}>$ $or\ a \in \sqrt{[<\bar{6}> + Soc(Z_{60}) :_z Z_{60}]} = \sqrt{[<\bar{2}> :_z Z_{60}]} = \sqrt{2Z} = 2Z.$

Remark (2.4)

Every weakly prime submodule of an R-module $\mathcal{H}$ is a wapp-primary submodule of $\mathcal{H}$ but not conversely.

Proof

Let E be a weakly prime submodule of $\mathcal{H}$, then by [1] E is weakly primary. Thus by remark (2.2) E is a wapp-primary. ■

Consider the following example for the converse

Example (2.5)

Let $\mathcal{H}$ be the Z-module Z_{60} and $E = <\bar{4}>$ be the submodule of Z_{60}. E is not weakly prime submodule of Z_{60}, since $0 \neq 2 \cdot \bar{2} \in E = <\bar{4}>$ $for\ 2 \in Z, \bar{2} \in Z_{60}$, but $\bar{2} \notin E = <\bar{4}>$ $and\ 2 \notin [<\bar{4}> :_z Z_{60}] = 4Z$. On the other hand $E = <\bar{4}>$ is a wapp-primary submodule of Z_{60} because whenever $0 \neq a \cdot \bar{h} \in E = <\bar{4}>$ $for\ a \in Z, \bar{h} \in Z_{60}$, implies that $\bar{h} \in E + Soc(Z_{60}) = <\bar{4}> + <\bar{2}> = <\bar{2}>$ $or \sqrt{[E + Soc(Z_{60}) :_z Z_{60}]} = \sqrt{[<\bar{4}> + <\bar{2}> :_z Z_{60}]} = \sqrt{[<\bar{2}> :_z Z_{60}]} = 2Z.$

The following are characterizations of wapp-primary submodule

Proposition (2.6)

A proper submodule E of an R-module $\mathcal{H}$ is wapp-primary if and only if whenever $0 \neq I \cdot F \subseteq E$. Where F is a submodule of $\mathcal{H}$, I is an ideal of R, implies that

$$F \subseteq E + Soc(\mathcal{H}) or\ I \subseteq \sqrt{[E + Soc(\mathcal{H}) \underset{R}{:} \mathcal{H}]}.$$

Proof

($\Longrightarrow$) Let $(0) \neq IF \subseteq E$ where I is an ideal of R, F is a submodule of $\mathcal{H}$ with $F \nsubseteq E + Soc(\mathcal{H})$, implies that there exists a nonzero $x \in F\ and\ x \notin E + Soc(\mathcal{H})$, we show that $I \subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Let $a \in I, if\ 0 \neq ax \in E$ and E is a wapp-primary, then $a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, that is $I \subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Thus, we assume that $ax = 0$. Now, suppose that $aF \neq (0)$, that is $0 \neq ac \in F$ for some $c \in F$. If $c \notin E + Soc(\mathcal{H})$ such that $c \notin E$. E is a wapp-primary , then $a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Thus $I \subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. If $c \in E \subseteq E + Soc(\mathcal{H})$, then $0 \neq ac \neq a(c + x) \in E$ and E is a wapp-primary, it follows that $c + x \in E + Soc(\mathcal{H})$ or $a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Thus $I \subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. So, we can assume that $aF = 0$. Now , suppose that $Ix \neq (0)$, that is $0 \neq sx \in E$ for some $s \in I$, and since E is a wapp-primary, then $s \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Now, since $0 \neq sx = (a + s)x \in E$ and E is a wapp-primary, so $a + s \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, that is $a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, hence $I \subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. So , we can assume that $Ix = (0)$. Since $IF \neq (0)$, implies that there exists $c_1 \in F, b \in I$ such that $0 \neq bc_1$ and $0 \neq bc_1 = b(c_1 + x) \in E$, thus we have two cases. CaseI: If b$\in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$ and $c_1 + x \notin E + Soc(\mathcal{H})$. But $0 \neq (a + b)(c_1 + x) = bc_1 \in$ E and E is a wapp-primary, then $(a + b) \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, hence a$\in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$, it follows that I$\subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. CaseII: If b$\notin \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$ and $c_1 + x \in E + Soc(\mathcal{H})$. Since $0 \neq bc_1 \in E$ and E is a wapp-primary we have $c_1 \in E + Soc(\mathcal{H})$ a contradiction. Hence I$\subseteq \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$.

($\Longleftarrow$) Suppose that $0 \neq bc_1 \in E$ for $a \in R, h \in \mathcal{H}$, then $(0) \neq <a><h> \subseteq E$, so by hypothesis $<h> \subseteq E + Soc(\mathcal{H})$ or $<a> \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. Hence $h \in E + Soc(\mathcal{H})$ or $a \in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$. That is E is a wapp-primary. ■

The following corollaries are direct application of proposition (2.6)

Corollary (2.7)

A proper sub module E of an R-module $\mathcal{H}$ is a wapp-primary if and only if whenever0≠aF⊆E. Where a∈R, F is a submodule of $\mathcal{H}$, implies that F⊆E+ $Soc(\mathcal{H})$ or a$\in \sqrt{[E + Soc(\mathcal{H}) :_R \mathcal{H}]}$.

Corollary (2.8)

A proper submodule E of an R-module $\mathcal{H}$ is a wapp-primary if and only if whenever (0) ≠Ih⊆E where I is an ideal of R and h∈$\mathcal{H}$, implies that h∈E+Soc($\mathcal{H}$) or I⊆$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$.

Proposition (2.9)

A proper submodule E of an R-module $\mathcal{H}$ is a wapp-primary if and only if whenever $[E:_R\mathcal{H}]\subseteq\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R h]$ for h∈$\mathcal{H}$-(E+Soc($\mathcal{H}$)).

Proof

(⟹) Let a∈$[E:_R h]$ where h∈$\mathcal{H}$-(E+Soc($\mathcal{H}$)), implies that ah∈E. If ah=0, then a∈$[0:_R h]$ so a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R h]$. If 0≠ah∈E and since E is a wapp-primary with h∉ E+Soc($\mathcal{H}$), implies that a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$, so a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R h]$. Thus $[E:_R h]\subseteq\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R h]$.

(⟸) Let 0≠ah∈E where a∈R, h∈$\mathcal{H}$ with h∉ $\mathcal{H}$-(E+Soc($\mathcal{H}$), then a∈$[E:_R h]$, it follows that a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R h]$. But 0≠ah, so a∉$[0:_R h]$, hence a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$. Thus E is a wapp-primary submodule of $\mathcal{H}$. ■

The following corollary is direct application of proposition (2.9).

Corollary (2.10)

A proper submodule E of an R-module $\mathcal{H}$ is a wapp-primary if and only if for a submodule F⊆$\mathcal{H}$-(E+Soc($\mathcal{H}$)), $[E:_R F]\subseteq\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}\cup[0:_R F]$.

Proposition (2.11)

A proper submodule E of an R-module $\mathcal{H}$ is a wapp-primary if and only if whenever a∈R and n is a positive integer $[E:_{\mathcal{H}} a]\subseteq[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]\cup[0:_{\mathcal{H}} a]$.

Proof

(⟹) Let h∈$[E:_{\mathcal{H}} a]$ with h∉ $E+Soc(\mathcal{H})$, implies that ah∈E. If ah=0, implies that h∈$[0:_{\mathcal{H}} a]$, so h∈$[E+Soc(\mathcal{H}):_R\mathcal{H}]\cup[0:_{\mathcal{H}} a]$. If 0≠ah∈E and E is a wapp-primary with h∉ $E+Soc(\mathcal{H})$,

then a^n∈ $[E+Soc(\mathcal{H}):_R\mathcal{H}]$ for some positive integer n. Thus $a^n\mathcal{H}\subseteq E+Soc(\mathcal{H})$, it follows that a^nh∈$E+Soc(\mathcal{H})$ for all h∈$\mathcal{H}$-($E+Soc(\mathcal{H})$), that is h∈$[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]$ and hence h∈$[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]$, so h∈$[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]\cup[0:_{\mathcal{H}} a]$. Thus $[E:_{\mathcal{H}} a]\subseteq[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]\cup[0:_{\mathcal{H}} a]$.

(⟸) Let 0≠ah∈E for a∈R, h∈$\mathcal{H}$ with h∉ $E+Soc(\mathcal{H})$. Since 0≠ah then h∉$[0:_{\mathcal{H}} a]$, implies that h∈$[E+Soc(\mathcal{H}):_{\mathcal{H}} a^n]$, that is a^nh∈$E+Soc(\mathcal{H})$ for all h∈$\mathcal{H}$-($E+Soc(\mathcal{H})$). a^n∈ $[E+Soc(\mathcal{H}):_R\mathcal{H}]$, implies that a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$. Thus E is a wapp-primary submodule of $\mathcal{H}$. ■

The following propositions show that under certain condition a proper submodule become wapp-primary.

Proposition (2.12)

Let $\mathcal{H}$ be an R-module, and E be a proper submodule of $\mathcal{H}$ with $[E+Soc(\mathcal{H}):_R\mathcal{H}]$ is a maximal semiprime ideal of R. Then E is a wapp-primary sub module of $\mathcal{H}$.

Proof

Assume that $[E+Soc(\mathcal{H}):_R\mathcal{H}]$ is a semiprime ideal of R, that is $\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}=[E+Soc(\mathcal{H}):_R\mathcal{H}]$. Let 0≠ah∈E for a∈R, h∈$\mathcal{H}$ with a∉ $\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}=[E+Soc(\mathcal{H}):_R\mathcal{H}]$. Since $[E+Soc(\mathcal{H}):_R\mathcal{H}]$ is a maximal, it follows that $\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$ is maximal, it follows that R=<a>+$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$, where <a> is an ideal of R, that is R=<a>+$[E+Soc(\mathcal{H}):_R\mathcal{H}]$, implies that 1=sa+b for some s∈R, b∈$[E+Soc(\mathcal{H}):_R\mathcal{H}]$. Hence h=sah+bh∈ $E+Soc(\mathcal{H})$, so E is a wapp- primary submodule of $\mathcal{H}$. ■

Proposition (2.13)

Let $\mathcal{H}$ be an R-module, and E be a proper sub module of $\mathcal{H}$ such that $[E+Soc(\mathcal{H}):_R\mathcal{H}]=[E+Soc(\mathcal{H}):_R F]$ for each submodule F of $\mathcal{H}$ with $E+Soc(\mathcal{H})$ ⊂F. Then E be a wapp-primary sub module of $\mathcal{H}$.

Proof

Let 0≠ah∈E for a∈R, h∈$\mathcal{H}$ with h∉ $E+Soc(\mathcal{H})$. Let F=($E+Soc(\mathcal{H})$)+<h> so $E+Soc(\mathcal{H})$ ⊂F, then h∈F, implies that a∈$[E:_R F]$ and E⊆$E+Soc(\mathcal{H})$, it follows that $[E:_R F]\subseteq[E+Soc(\mathcal{H}):_R F]$. But by hypothesis we have $[E+Soc(\mathcal{H}):_R F]=[E+Soc(\mathcal{H}):_R\mathcal{H}]$, it follows that $[E:_R F]\subseteq[E+Soc(\mathcal{H}):_R\mathcal{H}]$, implies that a∈$[E+Soc(\mathcal{H}):_R\mathcal{H}]\subseteq\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$. Thus a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$, that is E is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.14)

Let $\mathcal{H}$ be an R-module and $Soc(\mathcal{H})$ is a weakly primary submodule of $\mathcal{H}$. If E is a proper submodule of $\mathcal{H}$ with E⊆$Soc(\mathcal{H})$, then E is a wapp-primary submodule of $\mathcal{H}$.

Proof

Let 0≠ah∈E for a∈R, h∈$\mathcal{H}$. Since E⊆$Soc(\mathcal{H})$, implies that 0≠ah∈E⊆$Soc(\mathcal{H})$. But $Soc(\mathcal{H})$ is a weakly primary, then h∈$Soc(\mathcal{H})$ ⊆$E+Soc(\mathcal{H})$ or $a^n\mathcal{H}\subseteq Soc(\mathcal{H})$ ⊆$E+Soc(\mathcal{H})$. That is h∈$E+Soc(\mathcal{H})$ or a∈$\sqrt{[E+Soc(\mathcal{H}):_R\mathcal{H}]}$. Hence E is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.15)

Let $\mathcal{H}$ be an R-module, and A is a maximal semiprime ideal of R with A$\mathcal{H}$+$Soc(\mathcal{H})$ is a proper submodule of $\mathcal{H}$. Then A$\mathcal{H}$ is a wapp-primary submodule of $\mathcal{H}$.

Proof

Since A$\mathcal{H}$⊆ A$\mathcal{H}$+$Soc(\mathcal{H})$, then A⊆ $[A\mathcal{H}+Soc(\mathcal{H}):_R\mathcal{H}]$, that is there exists d∈$[A\mathcal{H}+Soc(\mathcal{H}):_R\mathcal{H}]$ and d∉A. But A is maximal, then we have R=A+<d>, thus 1=a+bd for some a∈A, b∈R, it follows that h=ah+bdh for each h∈$\mathcal{H}$, implies that h∈ $A\mathcal{H}+Soc(\mathcal{H})$ for each h∈$\mathcal{H}$. Thus, $\mathcal{H}\subseteq A\mathcal{H}+Soc(\mathcal{H})$, but $A\mathcal{H}+Soc(\mathcal{H})\subseteq\mathcal{H}$, it follows that $A\mathcal{H}+Soc(\mathcal{H})=\mathcal{H}$ a contradiction. Since $A\mathcal{H}+Soc(\mathcal{H})$ is a proper submodule of $\mathcal{H}$. Therefore d∈A and so that we have $[A\mathcal{H}+Soc(\mathcal{H}) :_R \mathcal{H}]\subseteq A$, it follows that $A=[A\mathcal{H}+Soc(\mathcal{H}) :_R \mathcal{H}]$ when is a maximal semiprime ideal of R. Hence by proposition (2.12) we have $A\mathcal{H}$ is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.16)

Let $\mathcal{H}$ be a torsionfree R-module and E be a nonzero sub module of $\mathcal{H}$with $Soc(\mathcal{H})\subseteq E$. Then following statements are equivalent:

1. E is a wapp-primary submodule of $\mathcal{H}$.
2. $[E \underset{\mathcal{H}}{:} A]$ is a wapp-primary submodule of $\mathcal{H}$ for every ideal A of R
3. $[E \underset{\mathcal{H}}{:} a]$ is a wapp-primary submodule of $\mathcal{H}$ for every a∈R.

Proof

(1)⟹(2): Let 0≠ah∈ $[E \underset{\mathcal{H}}{:} A]$, for a∈R, h∈$\mathcal{H}$, implies that a(hA)⊆E. If (0)≠ a(hA)⊆E, and E is a wapp-primary, implies that hA⊆ E+Soc($\mathcal{H}$) or $a\in\sqrt{[E+Soc(\mathcal{H}) :_R \mathcal{H}]}$. But $Soc(\mathcal{H})\subseteq E$, then E+Soc($\mathcal{H}$)=E, that is hA⊆E or $a\in\sqrt{[E :_R \mathcal{H}]}$, it follows that h∈ $[E \underset{\mathcal{H}}{:} A]$ or $a\in\sqrt{[E :_R \mathcal{H}]}\subseteq\sqrt{[[E \underset{\mathcal{H}}{:} A] :_R \mathcal{H}]}$. Thus, h∈ $[E \underset{\mathcal{H}}{:} A]\subseteq [E \underset{\mathcal{H}}{:} A]+Soc(\mathcal{H})$ or $a^n\mathcal{H}\subseteq[E \underset{\mathcal{H}}{:} A]\subseteq [E \underset{\mathcal{H}}{:} A]+Soc(\mathcal{H})$ for some positive integer n. Thus h∈$[E \underset{\mathcal{H}}{:} A]+Soc(\mathcal{H})$ or $a^n\in\sqrt{[[E \underset{\mathcal{H}}{:} A]+Soc(\mathcal{H}) :_R \mathcal{H}]}$. Hence $[E \underset{\mathcal{H}}{:} A]$ is a wapp-primary submodule of $\mathcal{H}$. If (0)=a(hA), then a(hb)=0 for some nonzero b∈A, implies that ah∈T($\mathcal{H}$). Since $\mathcal{H}$ is torsionfree, then T($\mathcal{H}$)=(0). Thus ah=0 a contradiction.

(2)⟹(3): Straight forward.

(3)⟹(1): It follows easily by taking a=1. ■

Proposition (2.17)

Let $\mathcal{H}$ be an R-module, and E is a proper submodule of $\mathcal{H}$ with $Soc(\mathcal{H})\subseteq E$. Then E is a wapp-primary if and only if $\frac{\mathcal{H}}{E}$ is a nonzero R-module and for every zero divisor s of $\frac{\mathcal{H}}{E}$ there exists h∈$\mathcal{H}$ and h∉E such that $s\in[0 :_R h]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$.

Proof

(⟹) Since s is a zero divisor an R-module $\frac{\mathcal{H}}{E}$, then there exists a nonzero element h+E∈$\frac{\mathcal{H}}{E}$ such that s(h+E)=E, that is sh+E=E, implies that sh∈E and h∉E. Since $Soc(\mathcal{H})\subseteq E$, then E+ $Soc(\mathcal{H})$=E, it follows that h∉E+ $Soc(\mathcal{H})$. If sh=0 , then $s\in[0 :_R h]$, implies that $s\in[0 :_R h]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$. If 0≠sh∈E and E is a wapp-primary, then $s^n\mathcal{H}\subseteq$ E+ $Soc(\mathcal{H})$ for some positive integer n . But $Soc(\mathcal{H})\subseteq E$, then E+ $Soc(\mathcal{H})$=E, that is $s^n\mathcal{H}\subseteq E$, that is $s^n\frac{\mathcal{H}}{E}$=(0), it follows that $s^n\in[0 :_R \frac{\mathcal{H}}{E}]$, so $s\in\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$. Therefore $s\in[0 :_R h]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$.

(⟸) Since $\frac{\mathcal{H}}{E}$ is a nonzero submodule, then E is a proper submodule of $\mathcal{H}$. Now, let 0≠sh∈E, s∈R, h∈$\mathcal{H}$ such that h∉E= E+ $Soc(\mathcal{H})$, it follows that s(h+E)=0=E for nonzero element h+E of $\frac{\mathcal{H}}{E}$. That is s is a zero divisor on $\frac{\mathcal{H}}{E}$, it follows that $s\in[0 :_R h]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$. But 0≠sh, implies that $s\notin[0 :_R h]$. So $s\in\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$, it follows that $s^n\frac{\mathcal{H}}{E}$=(0) for some positive integer n. That is $s^n\mathcal{H}\subseteq E\subseteq$ E+ $Soc(\mathcal{H})$, it follows that $s\in\sqrt{[E+Soc(\mathcal{H}) :_R \mathcal{H}]}$. Thus E is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.18)

Let $\mathcal{H}$ be an R-module, and E is a proper submodule of $\mathcal{H}$ with $Soc(\mathcal{H})\subseteq E$. If E is a wapp-primary submodule of $\mathcal{H}$ then E[x] (the set of all polynomial whose coefficients in E) is a wapp-primary submodule of an R-module $\mathcal{H}$[x].

Proof

Let g: $\mathcal{H}[x]\longrightarrow(\frac{\mathcal{H}}{E})[x]$ defined by $g(h_1x+h_2x^2+\ldots+h_nx^n)=(h_1+E)x+(h_2+E)x^2+\ldots+(h_n+E)x^n$ is an R-epimorphism, where $h_1,h_2,\ldots,h_n\in\mathcal{H}$ and $h_1+E, h_2+E,\ldots, h_n+E\in\frac{\mathcal{H}}{E}$. The kernel of g is obtain by reducing coefficients module E, implies that $\frac{\mathcal{H}[x]}{E[x]}\cong(\frac{\mathcal{H}}{E})[x]$. But $\frac{\mathcal{H}}{E}$ is a nonzero R-module, implies that $\frac{\mathcal{H}[x]}{E[x]}$ is a nonzero R-module. Now let s be a zero divisor on $(\frac{\mathcal{H}}{E})[x]$, then there exists a polynomial $(h_1+E)x+(h_2+E)x^2+\ldots+(h_n+E)x^n$ in $(\frac{\mathcal{H}}{E})[x]$ such that $s((h_1+E)x+(h_2+E)x^2+\ldots+(h_n+E)x^n)=(0)$. That is there exists 1≤j≤n such that $s(h_j+E)=(0)=E$ and $h_j+E\neq(0)=E$, implies that $s\,h_j\in E$ and $h_j\notin E+Soc(\mathcal{H})$. If $s\,h_j=0$ then $s\in[0 :_R h_j]$. If $0\neq s\,h_j\in E$ with $h_j\notin E+Soc(\mathcal{H})$ and E is a wapp-primary submodule of $\mathcal{H}$, implies that $s\in\sqrt{[E+Soc(\mathcal{H}) :_R \mathcal{H}]}$, that is $s^n\mathcal{H}\subseteq$ E+ $Soc(\mathcal{H})$. But $Soc(\mathcal{H})\subseteq E$ then E+ $Soc(\mathcal{H})$=E, it follows that $s^n\mathcal{H}\subseteq$ E, so $s^n\frac{\mathcal{H}}{E}$=(0), thus $s\in\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$. Hence $s\in[0 :_R h]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}]}$. Since $\frac{\mathcal{H}}{E}\subseteq(\frac{\mathcal{H}}{E})[x]$, it follows that $s\in[0 :_R h_i]\cup\sqrt{[0 :_R \frac{\mathcal{H}}{E}[x]]}$. Therefore by proposition (2.17) E[x] is a wapp-primary submodule of $\mathcal{H}$[x]. ■

Proposition (2.19)

Let $\mathcal{H}$ be an R-module, and E is a proper submodule of $\mathcal{H}$ with $[\mathcal{H} :_R F]\nsubseteq[\mathcal{H} :_R E+Soc(\mathcal{H})]$ and E+ $Soc(\mathcal{H})\subset$F for each submodule F of $\mathcal{H}$. If $[\mathcal{H} :_R E+Soc(\mathcal{H})]$ is a primary ideal of R, then E is a wapp-primary submodule of $\mathcal{H}$.

Proof

Let $0\neq ah\in E$, for $a\in R$, $h\in\mathcal{H}$, with $h\notin E+Soc(\mathcal{H})$. Since $E+Soc(\mathcal{H})$ is a proper submodule of $E+Soc(\mathcal{H})+\langle h\rangle=F$ and $[F :_R \mathcal{H}] \nsubseteq [E+Soc(\mathcal{H}) :_R \mathcal{H}]$, implies that there exists $s\in[F :_R \mathcal{H}]$ and $s\notin [E+Soc(\mathcal{H}) :_R \mathcal{H}]$, that is $s\mathcal{H}\subseteq F$ and $s\mathcal{H} \nsubseteq E+Soc(\mathcal{H})$. If $s\mathcal{H}\subseteq F$, then $r\,s\mathcal{H}\subseteq r(E+Soc(\mathcal{H})+\langle h\rangle)\subseteq E+Soc(\mathcal{H})$, it follows that $rs\in[E+Soc(\mathcal{H}) :_R \mathcal{H}]$. But $[E+Soc(\mathcal{H}) :_R \mathcal{H}]$ is a primary ideal of R and $s\notin [E+Soc(\mathcal{H}) :_R \mathcal{H}]$, then $r\in\sqrt{[E+Soc(\mathcal{H}) :_R \mathcal{H}]}$. Hence E is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.20)

Let $\mathcal{H}$ be an R-module, and G,F are proper submodule of $\mathcal{H}$ with $F\subseteq G$. If G is a wapp-primary submodule of $\mathcal{H}$, then $\frac{G}{F}$ is a wapp-primary submodule of $\frac{\mathcal{H}}{F}$.

Proof

Let $0\neq a(h+F)=ah+F\in\frac{G}{F}$, where $h\in\mathcal{H}$, $h+F\in\frac{G}{F}$, $a\in R$, it follows that $ab\in G$. If $ah=0$, then $a(h+F)=0$, gives a contradiction, thus $0\neq ah\in G$. Since G is a wapp-primary submodule of $\mathcal{H}$, then $h\in G+Soc(\mathcal{H})$ or $a^n\mathcal{H}\subseteq G+Soc(\mathcal{H})$, that is $h+F\in\frac{G+Soc(\mathcal{H})}{F}$ or $a^n\frac{\mathcal{H}}{F}\subseteq\frac{G+Soc(\mathcal{H})}{F}$, it follows that $h+F\in\frac{G}{F}+\frac{G+Soc(\mathcal{H})}{F}\subseteq\frac{G}{F}+Soc(\frac{\mathcal{H}}{F})$ or $a^n\frac{\mathcal{H}}{F}\subseteq\frac{G}{F}+\frac{G+Soc(\mathcal{H})}{F}\subseteq\frac{G}{F}+Soc(\frac{\mathcal{H}}{F})$. That is $h+F\in\frac{G}{F}+Soc(\frac{\mathcal{H}}{F})$ or $a\in\sqrt{[\frac{G}{F}+Soc(\frac{\mathcal{H}}{F}) :_R \frac{\mathcal{H}}{F}]}$. Hence $\frac{G}{F}$ is a wapp-primary submodule of $\frac{\mathcal{H}}{F}$. ■

Proposition (2.21)

Let $g\in Hom(\mathcal{H},\mathcal{H}')$ be an R-epimorphism and E is a submodule of $\mathcal{H}'$ such that $g^{-1}(E)$ is a wapp-primary submodule of $\mathcal{H}$. Then E is a is a wapp-primary submodule of $\mathcal{H}'$.

Proof

Let $g^{-1}(E)$ is a proper submodule of $\mathcal{H}$, then $g^{-1}(E)\neq\mathcal{H}$, that is there exists $h\in\mathcal{H}$ such that $h\notin g^{-1}(E)$, so $g(h)\notin E$, thus $E\neq\mathcal{H}$, so E is a proper submodule of $\mathcal{H}'$. Let $0\neq ah'\in E$ and $h'\notin E+Soc(\mathcal{H}')$, for $a\in R$, $h'\in\mathcal{H}'$. Since g is an epimorphism then there exists $h\in\mathcal{H}$ such that $g(h)=h'$, so $0\neq ah'=a\,g(h)=g(ah)\in E$. That is $0\neq ah\in g^{-1}(E)$ with $h\notin g^{-1}(E)+Soc(\mathcal{H})$. But $g^{-1}(E)$ is a wapp-primary submodule of $\mathcal{H}$, it follows that $a\in\sqrt{[g^{-1}(E)+Soc(\mathcal{H}) :_R \mathcal{H}]}$, that is $a^n\mathcal{H}\subseteq g^{-1}(E)+Soc(\mathcal{H})$. To show that $a^n\mathcal{H}'\subseteq E+Soc(\mathcal{H}')$. Let $h_1'\in\mathcal{H}'$ but since g is an epimorphism, then $g(h_1)=h_1'$ for some $h_1\in\mathcal{H}$. Thus $a^n h_1\in g^{-1}(E)+Soc(\mathcal{H})$, implies that $g(a^n h_1)=a^n g(h_1)=a^n h_1'\in g(g^{-1}(E)+g(Soc(\mathcal{H}))\subseteq E+Soc(\mathcal{H}')$. That is $a^n\in\sqrt{[E+Soc(\mathcal{H}') :_R \mathcal{H}']}$. That is $a^n\mathcal{H}\subseteq E+Soc(\mathcal{H}')$. Thus E is a wapp-primary submodule of $\mathcal{H}'$. ■

Proposition (2.22)

Let $g\in Hom(\mathcal{H},\mathcal{H}')$ be an R-monomorphism and E is a wapp-primary submodule of $\mathcal{H}'$. Then $g^{-1}(E)$ is a wapp-primary submodule of $\mathcal{H}$.

Proof

Let $0\neq ah\in g^{-1}(E)$ for $a\in R$, $h\in\mathcal{H}$ with $h\notin g^{-1}(E)+Soc(\mathcal{H})$, that is $0\neq g(ah)=ag(h)\in g(g^{-1}(E))=E$, it follows that $0\neq ag(h)\in E$. But E is a wapp-primary submodule of $\mathcal{H}'$, implies that $a^n\mathcal{H}'\subseteq E+Soc(\mathcal{H}')$ for some positive integer n. To show that $a^n g^{-1}(\mathcal{H}')\subseteq g^{-1}(E)+Soc(\mathcal{H})$. If $h_1\in g^{-1}(\mathcal{H}')$, then $g(h_1)\in\mathcal{H}'$. Thus $a^n g(h_1)=g(a^n h_1)\in E+Soc(\mathcal{H}')$, implies that $a^n h_1\in g^{-1}(E)+g^{-1}(Soc(\mathcal{H}'))$, it follows that $a^n g^{-1}(\mathcal{H}')\subseteq g^{-1}(E)+Soc(\mathcal{H})$. Thus $g^{-1}(E)$ is a wapp-primary submodule of $\mathcal{H}$. ■

Proposition (2.23)

Let $\mathcal{H}$ be an R-module, and G,F are submodules of $\mathcal{H}$ with $G\subseteq F$ and F is an essential submodule of $\mathcal{H}$. If G is a wapp-primary submodule of $\mathcal{H}$, then G is a wapp-primary submodule of F.

Proof

Let $0\neq ah\in G$, where $a\in R$, $h\in F\subseteq\mathcal{H}$, implies that $h\in\mathcal{H}$. Since G is a wapp-primary submodule of $\mathcal{H}$, then $h\in G+Soc(\mathcal{H})$ or $a\in\sqrt{[G+Soc(\mathcal{H}) :_R \mathcal{H}]}$. But F is an essential then by [5,p29] $Soc(E)=Soc(\mathcal{H})$. Thus $h\in$ $h\in G+Soc(F)$or $a\in\sqrt{[G+Soc(F) :_R \mathcal{H}]}\subseteq\sqrt{[G+Soc(F) :_R F]}$. Hence G is a wapp-primary submodule of F. ■

Proposition (2.24)

Let G and F be submodule of an R-module $\mathcal{H}$ with F is not contained in G and $Soc(\mathcal{H})\subseteq F$. If G is a wapp-primary submodule of $\mathcal{H}$, then $F\cap G$ is a wapp-primary submodule of F.

Proof

Since F is not contained in G, then $F\cap G$ is a proper submodule of G. Let $0\neq ah\in F\cap G$, for $a\in R$, $h\in F\subseteq\mathcal{H}$, it follows that $0\neq ah\in F$. Since G is a wapp-primary submodule of $\mathcal{H}$, implies that $h\in G+Soc(\mathcal{H})$ or $a^n\mathcal{H}\subseteq G+Soc(\mathcal{H})$, thus $h\in(G+Soc(\mathcal{H}))\cap F$ or $a^n\mathcal{H}\subseteq(G+Soc(\mathcal{H}))\cap F$ for some positive integer n. Since $Soc(\mathcal{H})\subseteq F$, it follows by modular law $h\in(F\cap G)+(F\cap Soc(\mathcal{H}))$ or $a^n\mathcal{H}\subseteq(F\cap G)+(F\cap Soc(\mathcal{H}))$. Then by [10, coro.9.9] we have $F\cap Soc(\mathcal{H})=Soc(F)$. Hence $h\in(F\cap G)+Soc(F)$ or $a^n\mathcal{H}\subseteq(F\cap G)+Soc(F)$ that is $a^n\subseteq[(F\cap G)+Soc(F) :_R \mathcal{H}]\subseteq[(F\cap G)+Soc(F) :_R F]$, it follows that $a^n F\subseteq(F\cap G)+Soc(F)$. Hence $F\cap G$ is a wapp-primary submodule of F. ■

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المقاسات الجزئية المتقاربة الابتدائية الضعيفة ومفاهيم ذات علاقة

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الملخص

لتكن حلقة R ابدالية بمحايد و $\mathcal{H}$ مقاسا احاديا ايسرا في هذا البحث قدمنا مفهوم المقاس الجزئي المتقارب الابتدائي الضعيف كأعمام جديد لمفهوم المقاس الجزئي الابتدائي الضعيف كذلك اعمام للمقاس الجزئي الاولي الضعيف. العديد من الخصائص الاساسية لهذا المفهوم درست. بالإضافة الى ذلك العديد من المكافئات ولامثلة لهذا المفهوم وجدت. اخيرا العديد من المكافئات لمفهوم المقاسات الجزئية المتقاربة الابتدائية الضعيفة في صف المقاسات الضربية أعطيت.