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On Some Differential Operators Related to A Type of Meromorphic Univalent Functions

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ABSTRACT

The goal of this article is to introduce a class of a type of meromorphic function which defined by some differential operators. Then we study various properties of these operators such as, coefficient inequality, both of growth and distortion theorems, radii of Starlikeness and convexity of $f(z)$ in the class that we present it.

حول بعض المؤثرات التفاضلية المتعلقة بنوع من الدوال أحادية التكافؤ

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الملخص

الهدف من هذه المقالة هو تقديم فئة من الدوال أحادية التكافؤ والتي تعرف بوساطة بعض المؤثرات التفاضلية. ثم درسنا خصائص مختلفة لهذه المؤثرات مثل عدم المساواة في المعامل، كلاً من مبرهنتي النمو والتشويه، ونصف قطر Starlikeness وتحدب $f(z)$ في الفئة التي قدمناها.

الكلمات المفتاحية: الأحادي، مؤثر، تشويه، نجمي، محدب، وظيفة.

Introduction

Let C be the complex plane. A function $f(z)$ is analytic at z_0 in a domain D if it is differentiable in some neighborhood of z_0, and it is analytic on a domain D if it is analytic at all points in D. A function $f(z)$ which is analytic on a domain D is said to be univalent there if it is a one-to-one mapping on D, and $f(z)$ is locallyunivalent at $z_0 \in$ D if it is univalent in some neighborhood of z_0. It is evident that $f(z)$ is locally univalent at z_0 provided $f'(z_0) \neq 0$.

Suppose $\sum$ denote the class of functions which be in the form

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$$f(z) = z^{-1} + \sum_{l=0}^{\infty} a_l z^l \qquad (1)$$

where $f(z)$ analytic in the punctured disk $\mathfrak{D} = \{z \in C: 0 < |z| < 1\}$. [1]

The function $f(z) \in \sum$ said to be meromorphic starlike if satisfy the following inequality:

$$Re\left(\frac{zf'(z)}{f(z)}\right) < 0 \qquad (2)$$

where $z \in \mathfrak{D}$. and the class of all meromorphic starlike functions will denoting by $\sum^*$

If $f \in \sum$ then we say that f is a meromorphic convex if the following inequality is holding:

$$R\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (3)$$

where $z \in \mathfrak{D}$.convolution

$$f(z) = z^{-1} + \sum_{l=0}^{\infty} a_l z^l$$

$$g(z) = z^{-1} + \sum_{l=0}^{\infty} b_l z^l$$

$$f * g = z^{-1} + \sum_{l=0}^{\infty} a_l b_l z^l = g * f$$

and the class of the above functions will be denoted by $\sum^C$.[2]

Osama et al. [3] studied the following function:

$$D_p^{\lambda}(z) = z^{-1} + \sum \left(\frac{\lambda}{n+1+\lambda}\right)^m z^l \qquad (4)$$

where $\lambda > 0, m > 0$.

In [4] Akanksha et al. studied the following linear operator:

$$D_{\delta}^{\lambda} f(z) = z^{-1} + \sum 1 + \delta(m+1)^n a_l z^l \qquad (5)$$

Where $n \in N = 0,1,2,\dots$.

We can generalize the above operator by using construction as

$$D_{p,\delta}^{\lambda} f(z) = \frac{1}{z} + \sum_{l=0}^{\infty} \frac{\lambda^m (1 + \delta(m+1)^n)}{(n+1+\lambda)^m} a_l z^l \qquad (6)$$

Let A, B be constants belong to $[-1,1]$. The function (1) exists in the class $\mathcal{T}_{\lambda}(p,\delta,A,B)$ if the following condition is satisfied:

$$\left|\frac{z\left(D_{p,\delta}^{\lambda} f(z)\right)' + D_{p,\delta}^{\lambda} f(z)}{Bz\left(D_{p,\delta}^{\lambda} f(z)\right)' + AD_{p,\delta}^{\lambda} f(z)}\right| < 1 \qquad (7)$$

$, z \in \mathfrak{B}$

$\forall z \in \mathfrak{B} = \{z: 0 < |z| < 1\}$.

Different classes and subclasses of $\sum$ have been studied by [5], and also [6,7,8]. [9] was the motivation to continue work of same studies. We will define and test some operators that satisfied some theorems related to meromorphic functions.

2. Definitions

Definition 2.1: Let $\mathcal{T}_{\lambda}(p,\delta,A,B)$ be a subclass of $\sum$ consists of functions of a form (1) which satisfy

$$\left|\frac{z\left(D_{p,\delta}^{\lambda} f(z)\right)' + D_{p,\delta}^{\lambda} f(z)}{Bz\left(D_{p,\delta}^{\lambda} f(z)\right)' + AD_{p,\delta}^{\lambda} f(z)}\right| < 1 \qquad (8)$$

Where $-1 \le A \le B \le 1, \delta \in N$ and p, λ are positive numbers.

3. Coefficient Inequality

The necessary and sufficient condition which make $f(z)$ in the class $\mathcal{T}_{\lambda}(p,\delta,A,B)$ can be obtained in the next theorem.

Theorem 3.1: Let a function f given in equation (1) belongs to $\sum$. Then f is in the class $\mathcal{T}_{\lambda}(p,\delta,A,B)$ if

$$\sum a_l \le \frac{(A-B)(n+1+\lambda)^m}{\lambda^m (1 + \delta(m+1)^n)\big(L(1-B) + (1-A)\big)} \qquad (9)$$

Proof: suppose $f \in \mathcal{T}_{\lambda}(p,\delta,A,B)$. So, we get from Eq. (9)

$$\left|\frac{z\left(-z^{-2} + \sum \frac{l[\lambda^m (1+\delta(m+1)^n)]}{(n+1+\lambda)^m} a_l z^{L-1} + z^{-1} + \sum \frac{\lambda^m (1+\delta(m+1)^n)}{(n+1+\lambda)^m}\right) a_l z^l}{Bz\left(-z^{-2} + \sum \frac{l[\lambda^m (1+\delta(m+1)^n)]}{(n+1+\lambda)^m} a_l z^{L-1} + A \sum \frac{\lambda^m (1+\delta(m+1)^n)}{(n+1+\lambda)^m}\right) a_l z^l}\right| < 1$$

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$$\Rightarrow\left|\frac{-z^{-1}\sum\frac{l[\lambda^m(1+\delta(m+1)^n)]}{(n+1+\lambda)^m}a_l z^L+z^{-1}+\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}a_l z^l}{Bz+B\sum\frac{l[\lambda^m(1+\delta(m+1)^n)]}{(n+1+\lambda)^m}a_l z^L+Az^{-1}+A\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}a_l z^l}\right|<1$$

$$\Rightarrow\left|\sum\frac{(l+1)[\lambda^m(1+\delta(m+1)^n)]}{(n+1+\lambda)^m}a_l z^l\right|$$
$$\leq\left|(A-B)z^{-1}\right.$$
$$\left.+(Bl+A)\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}a_l z^l\right|$$
$$\Rightarrow\sum\frac{(l+1)[\lambda^m(1+\delta(m+1)^n)]}{(n+1+\lambda)^m}a_l|z^l|$$
$$\leq(A-B)|z^{-1}|$$
$$+(Bl+A)\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}a_l z^l$$
$$\Rightarrow\left[\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}(l(1-B)+(1-A))a_l|z^l|\leq\frac{(A-B)}{|z|}\right]\cdot z$$
$$\Rightarrow\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}(l(1-B)+(1-A))a_l|z|^{l+1}\leq(A-B)$$

when $|z|=1$, we have

$$\sum\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}(l(1-B)+(1-A))a_l\leq(A-B)$$

$$\Rightarrow\sum a_l\leq\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)(L(1-B)+(1-A))}$$

Because $|Re(z)|\leq|z|,\forall z$, we get

$$R\left\{\frac{(l+1)[\lambda^m(1+\delta(m+1)^n)]}{(n+1+\lambda)^m}a_l z^l\right\}<1 \tag{10}$$

The sharp for $f(z)$ defined as

$$f(z)=\frac{1}{z}+\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n(L(1-B)+(1-A))}Z^l \tag{11}$$

4. Growth and Distortion Theorems

In the next theorem, we formulate some properties of growth and distortion of the function $f\in\mathcal{T}_\lambda(p,\delta,A,B)$.

Theorem 4.1: If f be in the form equation (1) and $f\in\mathcal{T}_\lambda(p,\delta,A,B)$. If with substitute $|z|=r>0$, then we obtain

$$\frac{1}{r}-\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)+(1-A)\big)}r\leq|f(z)|\leq\frac{1}{r}+\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(L(1-B)+(1-A)\big)}r$$

Proof: Suppose $f\in\mathcal{T}_\lambda(p,\delta,A,B)$, from theorem 3.1 we have:

$$\frac{1}{|z|}-\sum a_l|z|^l\leq|f(z)|\leq\frac{1}{|z|}+\sum a_l|z|^l$$

$$\Rightarrow\frac{1}{|z|}-\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(L(1-B)+(1-A)\big)}|z|^l\leq|f(z)|\leq\frac{1}{|z|}+\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)+(1-A)\big)}|z|^l$$

When $l=1$

$$\frac{1}{|z|}-\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big((1-B)+(1-A)\big)}|z|\leq|f(z)|\leq\frac{1}{|z|}+\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big((1-B)+(1-A)\big)}|z|$$

Substitute $|z|=r$ to get

$$\frac{1}{r}-\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)+(1-A)\big)}r\leq|f(z)|\leq\frac{1}{r}+\frac{(A-B)(n+1+\lambda)^m}{\lambda^m(1+\delta(m+1)^n)\big(L(1-B)+(1-A)\big)}r$$

Theorem 4.2: If f be in the form (1) and $f\in\mathcal{T}_\lambda(p,\delta,A,B)$. If $|z|=r>0$, then

$$\frac{1}{r^2}-\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}\leq|f(z)'|\leq\frac{1}{r^2}+\frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}$$

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Proof: Suppose $f \in \mathcal{T}_\lambda(p,\delta,A,B)$. Define the following derivative:

$$\frac{d}{dz}\left(\frac{1}{z} - \sum a_l z^l\right) \le |f(z)'| \le \frac{d}{dz}\left(\frac{1}{z} + \sum a_l z^l\right)$$

By referring to theorem 3.1, we find

$$\frac{1}{|z|^2} - \frac{l\left(\lambda^m(1+\delta(m+1)^n)\right)}{(n+1+\lambda)^m}\Bigg| |z|^{l-1} \le |f(z)'| \le \frac{1}{|z|^2} + \frac{L\left(\lambda^m(1+\delta(m+1)^n)\right)}{(n+1+\lambda)^m}\Bigg| |z|^{l-1}$$

Substitute $l = 1$

$$\Rightarrow \frac{1}{|z|^2} - \frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m} \le |f(z)'| \le \frac{1}{|z|^2} + \frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}$$

And when $|z| = r$

$$\Rightarrow \frac{1}{r^2} - \frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m} \le |f(z)'| \le \frac{1}{r^2} + \frac{\lambda^m(1+\delta(m+1)^n)}{(n+1+\lambda)^m}$$

5. Radii at Starlikeness and Convexity

We can obtain a radiiof starlike and convex for $\mathcal{T}_\lambda(p,\delta,A,B)$ by the following theorems.

Theorem 5.1: It is said that $f(z) \in \mathcal{T}_\lambda(p,\delta,A,B)$ of the form (1). Then f is meromorphically starlike of order λ, $0 \le \lambda < 1$ in $|z| < r_1(p,\delta,A,B)$

where

$$r_1(p,\delta,A,B) = \inf_{l\ge1}\left|\frac{\lambda^m(1+\delta(m+1)^n(l(1-B)) + (1-A)(1-\lambda)}{(A-B)(n+1+\lambda)^m(l+2-\lambda)}\right|^{\frac{1}{l+1}}$$

And f of the form equation (1).

Proof: First, we will prove that $\left|\frac{zf\prime(z)}{f(z)} + 1\right| \le 1 - \lambda$

$$\left|\frac{zf'(z)}{f(z)} + 1\right| = \left|\frac{zf'(z)}{f(z)} + \frac{f(z)}{f(z)}\right| = \left|\frac{z(-z^2 + \sum La_l z^{l-1}) + z^{-1} + \sum a_l z^l}{z^{-1} + \sum a_l z^l}\right|$$

$$= \left|\frac{-z^{-1} + \sum La_l z^l + z^{-1} + \sum a_l z^l}{z^{-1} + \sum a_l z^l}\right| = \left[\left|\frac{\sum(l+1)a_l z^l}{z^{-1} + \sum a_l z^l}\right|\right] \cdot z$$

$$= \left|\frac{\sum(l+1)a_l z^{l+1}}{1 + \sum a_l z^{l+1}}\right|$$

$\because \left|\frac{z_1+z_2}{z_1+z_2}\right| \le \frac{|z_1|+|z_2|}{|z_1|-|z_2|}$ (From properties of absolute value)

$$\therefore \left|\frac{\sum(l+1)a_l z^{l+1}}{1 + \sum a_l z^{l+1}}\right| \le \frac{\sum(l+1)a_i |z|^{l+1}}{1 - \sum a_i |z|^{l+1}}$$

$$\therefore \frac{\sum(l+1)a_l z^{l+1}}{1 + \sum a_l z^{l+1}} \le 1 - \lambda$$

$$\sum(l+1)a_l|z|^{l+1} \le (1-\lambda)(1 - \sum a_l |z|^{l+1})$$

$$\sum(l+1)a_l|z|^{l+1} \le 1 - \sum a_i|z|^{l+1} - \lambda + \lambda \sum a_i |z|^{l+1}$$

$$\sum(l+1)a_l|z|^{l+1} \le \sum a_l|z|^{l+1}(\lambda-1) - \lambda + 1$$

$$\sum(l+1)a_l|z|^{l+1} - \sum(\lambda-1)a_l|z|^{l+1} \le (1-\lambda)$$

$$\sum(l+1) + (1-\lambda)a_l|z|^{l+1} \le (1-\lambda)$$

$$\sum(l+2-\lambda)a_l|z|^{l+1} \le (1-\lambda) \qquad (12)$$

$$\sum a_l \le \frac{(A-B)(n+1+\lambda)^m}{\lambda^n(1+\delta(m+1)^n)\left(l(1-B)\right) + (1-A)}$$

By using equation (12)

$$\frac{(A-B)(n+1+\lambda)^m(l+2-\lambda)}{\lambda^m(1+\delta(m+1)^n(l(1-B)) + (1-A)}|Z|^{l+1} \le (1-\lambda)$$

$$|Z|^{l+1} \le \frac{\lambda^m(1+\delta(m+1)^n(l(1-B)) + (1-A)(1-\lambda)}{(A-B)(n+1+\lambda)^m(l+2-\lambda)}$$

$$|z| \le \left|\frac{\lambda^m(1+\delta(m+1)^n(l(1-B)) + (1-A)(1-\lambda)}{(A-B)(n+1+\lambda)^m(1+2-\lambda)}\right|^{\frac{1}{l+1}}$$

$$= \inf_{L\ge1}\left|\frac{\lambda^m(1+\delta(m+1)^n(l(1-B)) + (1-A)(1-\lambda)}{(A-B)(n+1+\lambda)^m(l+2-\lambda)}\right|^{\frac{1}{l+1}}$$

Theorem 5.2: It is said that $f(z) \in \mathcal{T}_\lambda(p,\delta,A,B)$ of the form equation (1). Then f is

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meromorphically convex of order α, $0 \le \alpha < 1$ in $|z| < r_2(p, \delta, A, B)$

Where

$$r_2(p,\delta,A,B) = \inf_{l\ge 1}\left\{\frac{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)\big)+(1-A)(1-\alpha)}{(A-B)(n+1+\lambda)^m(l+2-\alpha)}\right\}^{\frac{1}{l+1}}$$

And f of the form equation (1).

Proof: First, we will prove that $\left|\frac{zf''(z)}{f'(z)}+2\right| \le 1-\alpha$

$$\left|\frac{2f''(z)+zf'(z)}{f'(z)}\right| = \left|\frac{2z^{-2}+\sum l(l-1)a_l z^l - 2z^{-2} + 2\sum l a_l z^{l+1}}{-z^{-2}+\sum l a_l z^{l-1}}\right|$$

$$= \left|\frac{\sum l(l-1+2)a_l z^{l-1}}{-z^{-2}+\sum l a_l z^{l-1}}\right|$$

$$= \left|\frac{\sum l(l-1)a_l z^{l-1}}{-z^{-2}+\sum l a_l z^{l-1}}\right| \cdot z^2$$

$$= \left|\frac{\sum l(l+1)a_l z^{l+1}}{-1+\sum l a_l z^{l+1}}\right| \le \left|\frac{\sum l(l+1)a_l z^{l+1}}{1-\sum l a_l z^{l+1}}\right| \le (1-\alpha)$$

$$\sum l(l+1)a_l|z|^{l+1} \le \left(1-\sum l a_l z^{l+1}\right)(1-\alpha)$$

$$\sum l(l+1)\, a_l|z|^{l+1} \le 1-\sum l a_l z^{l+1} - \alpha + \alpha\sum l a_l z^{l+1}$$

$$\sum l(l+1)\, a_l|z|^{l+1} + \sum l a_l z^{l+1} - \alpha\sum l a_l z^{l+1} \le (1-\alpha)$$

$$\sum l(l+1+1-\alpha)\, a_l|z|^{l+1} \le (1-\alpha)$$

$$\sum (l+2-\alpha)\, a_l|z|^{l+1} \le (1-\alpha)$$

$$\frac{(A-B)(n+1+\lambda)^m(l+2-\alpha)}{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)\big)+(1-A)}|z|^{l+1} \le (1-\alpha)$$

$$|z|^{l+1} \le \frac{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)\big)+(1-A)(1-\alpha)}{(A-B)(n+1+\lambda)^m(l+2-\alpha)}$$

$$|Z|^{l+1} \le \left|\frac{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)\big)+(1-A)(1-\alpha)}{(A-B)(n+1+\lambda)^m(L+2-\alpha)}\right|^{\frac{1}{l+1}}$$

$$\inf_{l\ge 1}\left\{\frac{\lambda^m(1+\delta(m+1)^n)\big(l(1-B)\big)+(1-A)(1-\alpha)}{(A-B)(n+1+\lambda)^m(l+2-\alpha)}\right\}^{\frac{1}{l+1}}$$

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