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Estimates of Coefficients for Bi-Univalent Functions in the Subclass $\mathcal{H}_{\Sigma}(n,\gamma,\varphi)$

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ABSTRACT

Considering that finding the bounds for the coefficients of the Taylor-Maclaurin series expansion of bi-univalent functions is one of the important subjects in geometric function theory that has attracted the attention of many researchers in the last few decades, we also take a step in this direction. Finding such bounds is the main focus or, more clearly, the main problem of our work. In this article, we study the subclass $\mathcal{H}_{\Sigma}(n,\gamma,\varphi)$ of bi-univalent functions which is defined in the open unit disk D. Furthermore, we obtained the upper bounds estimates for the first coefficients $|a_2|$ and $|a_3|$ of the functions in this category by using subordination method. From the main result of the article (Theorem 2.1), special cases have been derived that improve some the results of previous articles.

تقديرات لمعاملات الدوال ثنائية التكافؤ في الفئة الجزئية $\mathcal{H}_{\Sigma}(n,\gamma,\varphi)$

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الملخص

يعد إيجاد حدود لمعاملات توسيع متسلسلة تايلور – ماكلورين للدوال ثنائية التكافؤ من المواضيع المهمة نظرية الدالة الهندسية التي جذبت انتباه العديد من الباحثين في العقود القليلة الماضية، فقد اتخذنا نحن أيضًا خطوة في هذا الاتجاه بإيجاد مثل هذه الحدود بالتركيز الأساس وبوضوح أكثر المسألة الرئيسة لعملنا في هذا البحث هي دراسة الفئة الجزئية $\mathcal{H}_{\Sigma}(n,\gamma,\varphi)$ للدوال ثنائية التكافؤ والمعرفة في القرص المفتوح D . فضلاً عن ذلك حصلنا على تقديرات للمعاملات الأولى $|a_2|$ و $|a_3|$ للدوال في هذا الصنف باستخدام طريقة التبعية. ومن النتيجة الرئيسة لهذا البحث (مبرهنة 2−1)، تم اشتقاق حالات خاصة تطور بعض النتائج التي تم الحصول عليها في بحوث سابقة.

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Introduction

In [1] carried out research on the category of bi-univalent functions and demonstrated that $|a_2| \leq 1.51$ for every bi-univalent function f which in turn resulted in further research into the bounds of bi-univalent functions' coefficients. Then, [2] displayed $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$ in 1969. Afterwards, in [3] proved that $|a_2| \leq \sqrt{2}$. [4] has recently revived interest in the research of bi-univalent functions and offered a spearheading work in this regard in 2010. Several different subclasses of the bi-univalent functions category have been presented and investigated similarly by a large number of authors who followed the project of Srivastava et al. in a substantial number of works. A number of authors investigated categories of bi-univalent holomorphic functions and discovered estimation of the coefficient's estimation issue for any of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these categories [5, 6]. Also, they provided some fascinating instances of characterization and functions of this category. Another exciting and new work is done by [7]. They obtained the coefficient estimates for a new subclass $\mathcal{G}_q^{\beta}(H, 2u, v)$ of analytic functions that is defined by quasi-subordination and the other exciting and new work is done by [8] they obtained the coefficient estimates for a new subcategory $\mathcal{H}_{\Sigma}(n, \beta, \emptyset)$ of analytic functions that is defined by subordination.

Using the method of convolution on the category of holomorphic functions which is defined on the open unit disk $\mathcal{D} = \{z \in \mathbb{C};\ |z| < 1\}$, [9] indicated the operator $\mathcal{R}$ as follows:

$$\mathcal{R}^{\lambda} f(z) = f(z) * \frac{z}{(1 - z)^{\lambda+1}},$$

where, f is a holomorphic function which is defined on the open unit disk $\mathcal{D}$, $z \in \mathcal{D}$, $\lambda \in \mathbb{R}$ and $\lambda > -1$.

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get

$$\mathcal{R}^{n} f(z) = z\frac{\left(z^{n-1} f(z)\right)^{(n)}}{n!}.$$

The expression $\mathcal{R}^{n} f(z)$ is said to be an n^{th}-order Ruscheweyh derivative of $f(z)$ and the sign $*$ is used for Hadamard product (or convolution). We can deduce that [10]

$$\mathcal{R}^{n} f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k,$$

where,

$$\sigma(n,k) = \frac{\Gamma(n+k)}{(k-1)!\,\Gamma(n+1)}.$$

In this article and inspired by the mentioned works, we first define a subclass of bi-univalent functions in the open unit disk $\mathcal{D}$ by using the Ruscheweyh operator and subordination and then we study this subclass. One of the most important results of this article is finding an upper bound for the coefficients $|a_2|$ and $|a_3|$ of the functions of this category, this result is presented in Theorem 2.1 and some corollaries that are deduced from this theorem.

Suppose that $\mathbb{C}$ is the set of all complex numbers and $\mathcal{A}$ denotes the category of the functions f that are holomorphic in the open unit disk $\mathcal{D}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$.

To put it in another way, the Taylor-Maclaurin series expansion of the function f in $\mathcal{A}$ is a special

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form of the power series, which can be stated as follows:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{D} \qquad (1.1)$$

where, $a_k = \frac{f^{(k)}(0)}{k!}$; $k = 2, 3, 4, \cdots$ and we note that $a_1 = 1$.

Definition 1.1. [11] A function $f: \mathcal{D} \longrightarrow \mathbb{C}$ is called univalent function in $\mathcal{D}$ if it is holomorphic and injective (one-to-one) in $\mathcal{D}$, that is,

$$f(z_1) \neq f(z_2), \quad (z_1, z_2 \in \mathcal{D}, \quad z_1 \neq z_2).$$

In this paper, we will indicate the category of all functions that are univalent in $\mathcal{D}$ by $\mathcal{S}$ which is a subclass of $\mathcal{A}$ introduced by [11, 12].

For two holomorphic functions f and g in $\mathcal{D}$, we state that the function f is subordinate to the function g, and we write $f(z) \prec g(z)$, if there is a Schwarz function ω (i.e., a function that is holomorphic in $\mathcal{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathcal{D}$), such that $f(z) = g(\omega(z))$ for any z belonging to $\mathcal{D}$. Specifically, let the function g is univalent in $\mathcal{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subseteq g(\mathcal{D})$ (see [13]).

The familiar Koebe-one-quarter theorem [11] asserts that for any univalent function f in $\mathcal{S}$, the image of the open unit disk $\mathcal{D}$ under f includes a disk with radius $\frac{1}{4}$. Consequently, each univalent function f has an inverse f^{-1}, such that:

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{D}) \quad \text{and}$$

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f): r_0(f) \geq \frac{1}{4}\right).$$

In some disk, the following form is the inverse of the series expansion of the

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k \qquad (1.2)$$

function f about the origin:

The best bound for all $|b_k|$ in (1.2) is provided by the inverse of the Koebe function (see [14]).

We know, near the origin, every univalent function $f(z)$ and its inverse $f^{-1}(w)$ satisfy:

$$f(f^{-1}(w)) = w$$

in other words,

$$w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \cdots,$$

$$f^{-1}(w) = w - a_2[f^{-1}(w)]^2 - a_3[f^{-1}(w)]^3 - \cdots \qquad (1.3)$$

or by substituting (1.2) in (1.3), we obtain

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \qquad (1.4)$$

Utilized the idea of subordination to define the subcategories of convex and starlike functions by [15]. In this case, we assume that the holomorphic function φ has a non-negative real part in $\mathcal{D}$, $\varphi(\mathcal{D})$ is symmetric about the real axis, $\varphi(0) = 1$, $\varphi'(0) = J_1 > 0$, and φ is of the form of power series extension

$$\varphi(z) = 1 + J_1 z + J_2 z^2 + J_3 z^3 + \cdots; \quad z \in \mathcal{D} \qquad (1.5)$$

In [15], introduced the following categories of starlike and convex functions:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \varphi(z), \ z \in \mathcal{D} \right\},$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A}; 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad z \in \mathcal{D} \right\}.$$

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For $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, the categories $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ will be decrease to Janowski starlike and Janowski convex functions categories $\mathcal{S}^*[A,B]$ and $\mathcal{C}[A,B]$, respectively. It's worth noting that if $0 \leq \delta < 1$, then $\mathcal{S}^*[1-2\delta,-1] = \mathcal{S}^*(\delta)$, namely, the category of starlike functions of order δ and $\mathcal{C}[1-2\delta,-1] = \mathcal{C}(\delta)$, namely, the category of convex functions of order δ. In particular, the popular categories of convex and starlike functions in $\mathcal{D}$ are $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$, respectively. Furthermore, the characteristics of the category $\mathcal{S}_e^* = \mathcal{S}^*(e^z)$ were investigated by [16].

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathcal{D}$ if both f and f^{-1} are univalent functions in $\mathcal{D}$. Assume that $\sum$ denotes the category of all bi-univalent functions in $\mathcal{D}$, presented by the Taylor–Maclaurin series extension (1.1). The followings are instances of bi-univalent functions in $\sum$ [17]:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Nevertheless, the well-known Koebe function $\frac{z}{(1-z)^2}$ and the functions

$$z - \frac{1}{2}z^2, \quad \frac{z}{1-z^2}$$

which are the members of $\mathcal{S}$, are not included in $\sum$.

The goal of this paper is to derive bounds for the Taylor-Maclaurin series coefficients $|a_2|$ and $|a_3|$ of every function f in the subclass $\mathcal{H}_\Sigma(n,\gamma,\varphi)$ of bi-univalent functions. Also, the bounds for the first two coefficients of f^{-1} are given. A distinct technique involving subordination has been used to investigate the object of the article. We begin by introducing the category $\mathcal{H}_\Sigma(n,\gamma,\varphi)$ where $\sum$ represents the set of all bi-univalent functions in $\mathcal{D}$, n is the order of the Ruscheweyh operator of $f(z)$, γ is the nonnegative coefficient of the second order derivative of Ruscheweyh operator of $f(z)$ and φ is the holomorphic function of Ma and Minda type given by (1.5).

Definition 1.2. A function $f \in \sum$ given by (1.1) is called in the category $\mathcal{H}_\Sigma(n,\gamma,\varphi)$ if

$$\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)'' \prec \varphi(z); \quad z \in \mathcal{D},$$

and

$$\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)'' \prec \varphi(w); \quad w \in \mathcal{D},$$

where $\gamma \geq 0$, $g = f^{-1}$ and φ is the function given by (1.5).

Now, if $\mathcal{P}$ represents the family of all functions $q(z)$, which are holomorphic in $\mathcal{D}$ such that $q(0) = 1$ and $Re\left(q(z)\right) > 0$ $(z \in \mathcal{D})$ and they have the series expansion of the form:

$$q(z) = 1 + \sum_{i=1}^{\infty} c_n z^n \qquad (1.6)$$

The following outcome, which is known as Caratheodory's lemma, is required to determine the results of the paper.

Lemma 1.3.[16] If q belongs to $\mathcal{P}$, with $q(z)$ given by (1.6), then $|c_n| \leq 2$ for $n \geq 1$.

2. Results and Discussion

Theorem 2.1. If $f \in \mathcal{H}_\Sigma(n,\gamma,\varphi)$, then

$$|a_2| \leq \sqrt{\frac{J_1(J_1 + |J_2|)}{3\,J_1(1+2\gamma)\sigma(n,3) + 4(1+\gamma)^2\left(\sigma(n,2)\right)^2}}$$

and

$$|a_3| \leq \frac{J_1(J_1 + |J_2|)}{3\,J_1(1+2\gamma)\sigma(n,3) + 4(1+\gamma)^2\left(\sigma(n,2)\right)^2} + \frac{J_1}{3(1+2\gamma)\sigma(n,3)}.$$

Proof. Suppose that $f \in \mathcal{H}_\Sigma(n,\gamma,\varphi)$ and $g = f^{-1}$, then by definition of subordination there are Schwarz functions $u, v: \mathcal{D} \to \mathcal{D}$, such that

$$\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)'' = \varphi\left(u(z)\right); \quad z \in \mathcal{D},$$

and

$$\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)'' = \varphi\left(v(w)\right); \quad w \in \mathcal{D}.$$

Define the functions q_1 and q_2 by

$q_1(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\cdots$ and

$q_2(z)=\frac{1+v(z)}{1-v(z)}=1+d_1z+d_2z^2+\cdots.$

Or, in other words,

$$u(z)=\frac{q_1(z)-1}{q_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right)$$

and

$$v(z)=\frac{q_2(z)-1}{q_2(z)+1}=\frac{1}{2}\left(d_1z+\left(d_2-\frac{d_1^2}{2}\right)z^2+\cdots\right).$$

Then q_1 and q_2 are holomorphic in $\mathcal{D}$ with $q_1(0)=1=q_2(0)$. Now, since $u,v:\mathcal{D}\to\mathcal{D}$, the functions q_1 and q_2 have a non-negative real part in $\mathcal{D}$, and by Caratheodory's lemma $|c_i|\le 2$ and $|d_i|\le 2$ for each $i\in\mathbb{N}$. Since

$$\left(\mathcal{R}^nf(z)\right)'=1+2\,\sigma(n,2)\,a_2z+3\,\sigma(n,3)\,a_3z^2+\cdots,$$

and

$$\gamma z\left(\mathcal{R}^nf(z)\right)''=2\,\gamma\sigma(n,2)\,a_2z+6\,\gamma\sigma(n,3)\,a_3z^2+\cdots,$$

so,

$$\left(\mathcal{R}^nf(z)\right)'+\gamma z\left(\mathcal{R}^nf(z)\right)''=1+2(1+\gamma)\,\sigma(n,2)a_2z+3(1+2\gamma)\sigma(n,3)a_3z^2+\cdots. \quad (2.1)$$

On the other hand

$$\varphi(u(z))=\varphi\left(\frac{q_1(z)-1}{q_1(z)+1}\right)=1+\frac{1}{2}J_1c_1z+\left(\frac{1}{2}J_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}J_2c_1^2\right)z^2+\cdots. \quad (2.2)$$

Again, by using (1.4) we have

$$\mathcal{R}^ng(w)=w-\sigma(n,2)a_2w^2+\sigma(n,3)(2a_2^2-a_3)w^3+\cdots,$$

$$\left(\mathcal{R}^ng(w)\right)'=1-2\sigma(n,2)a_2w+3\sigma(n,3)(2a_2^2-a_3)w^2+\cdots,$$

and

$$\gamma w\left(\mathcal{R}^ng(w)\right)''=-2\gamma\sigma(n,2)a_2w+6\gamma\sigma(n,3)(2a_2^2-a_3)w^2+\cdots.$$

So,

$$\left(\mathcal{R}^ng(w)\right)'+\gamma w\left(\mathcal{R}^ng(w)\right)''=1-2\,(1+\gamma)\sigma(n,2)a_2w+3(1+2\gamma)\sigma(n,3)(2a_2^2-a_3)w^2+\cdots. \quad (2.3)$$

On the other hand

$$\varphi(v(w))=\varphi\left(\frac{q_2(w)-1}{q_2(w)+1}\right)=1+\frac{1}{2}J_1d_1w+\left(\frac{1}{2}J_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}J_2d_1^2\right)w^2+\cdots. \quad (2.4)$$

Now, since

$$\left(\mathcal{R}^nf(z)\right)'+\gamma z\left(\mathcal{R}^nf(z)\right)''=\varphi\left(\frac{q_1(z)-1}{q_1(z)+1}\right)$$

then, (2.1) and (2.2) yield

$$1+2(1+\gamma)\sigma(n,2)a_2z+3(1+2\gamma)\sigma(n,3)a_3z^2+\cdots=1+\frac{1}{2}J_1c_1z+\left(\frac{1}{2}J_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}J_2c_1^2\right)z^2+\cdots.$$

Again, since

$$\left(\mathcal{R}^ng(w)\right)'+\gamma w\left(\mathcal{R}^ng(w)\right)''=\varphi\left(\frac{q_2(w)-1}{q_2(w)+1}\right)$$

then, from (2.3) and (2.4) it follows that

$$1-2(1+\gamma)\sigma(n,2)a_2w+3(1+2\gamma)\sigma(n,3)(2a_2^2-a_3)w^2+\cdots=1+\frac{1}{2}J_1d_1w+\left(\frac{1}{2}J_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}J_2d_1^2\right)w^2+\cdots.$$

Therefore,

$$2\,(1+\gamma)\sigma(n,2)a_2=\frac{1}{2}J_1c_1, \quad (2.5)$$

$$3(1+2\gamma)\sigma(n,3)a_3=\frac{1}{2}J_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}J_2c_1^2 \quad (2.6)$$

$$2\,(1+\gamma)\sigma(n,2)a_2=-\frac{1}{2}J_1d_1 \quad (2.7)$$

and

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$$3(1+2\gamma)\sigma(n,3)(2a_2^2-a_3)$$
$$=\frac{1}{2}J_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}J_2d_1^2,$$

or

$$6(1+2\gamma)\sigma(n,3)a_2^2-3(1+2\gamma)\sigma(n,3)a_3 =\frac{1}{2}J_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}J_2d_1^2 \qquad (2.8)$$

From (2.5) and (2.7) we get

$$c_1=-d_1 \qquad (2.9)$$
$$8(1+\gamma)^2(\sigma(n,2))^2a_2^2=\frac{1}{2}J_1^2d_1^2 \qquad (2.10)$$

Now, apply (2.6), (2.9) and (2.10) on (2.8) to obtain

$$6(1+2\gamma)\sigma(n,3)a_2^2 =\frac{1}{2}J_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}J_2c_1^2 +\frac{1}{2}J_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}J_2d_1^2.$$

Then,

$$6(1+2\gamma)\sigma(n,3)a_2^2+\frac{1}{2}J_1d_1^2 =\frac{1}{2}J_1(c_2+d_2)+\frac{1}{2}J_2d_1^2,$$

and therefore,

$$a_2^2 =\frac{J_1^2(c_2+d_2)+J_1J_2d_1^2}{12\,J_1(1+2\gamma)\sigma(n,3)+16(1+\gamma)^2(\sigma(n,2))^2},$$

In this step we have

$$|a_2|^2 \le\frac{J_1^2|c_2+d_2|+J_1J_2|d_1|^2}{4[3\,J_1(1+2\gamma)\sigma(n,3)+4(1+\gamma)^2(\sigma(n,2))^2]}.$$

By Lemma 1.3

$$|a_2|^2 \le\frac{4\,J_1^2+4\,J_1\,|J_2|}{4\left[3\,J_1(1+2\gamma)\sigma(n,3)+4(1+\gamma)^2\big(\sigma(n,2)\big)^2\right]}$$

and hence

$$|a_2| \le\sqrt{\frac{J_1(J_1+\,|J_2|)}{3\,J_1(1+2\gamma)\sigma(n,3)+4(1+\gamma)^2\big(\sigma(n,2)\big)^2}}.$$

Next, in order to find the upper bound for $|a_3|$, by subtracting (2.8) from (2.6) and further computations lead to

$$6(1+2\gamma)\sigma(n,3)a_3 =6(1+2\gamma)\sigma(n,3)a_2^2+\frac{1}{2}J_1(c_2-d_2),$$

which gives

$$a_3=\frac{6(1+2\gamma)\sigma(n,3)a_2^2+\frac{1}{2}J_1(c_2-d_2)}{6(1+2\gamma)\sigma(n,3)},$$

and these yields

$$a_3=a_2^2+\frac{J_1(c_2-d_2)}{12\,(1+2\gamma)\sigma(n,3)}.$$

It follows that

$$|a_3|\le\frac{J_1(J_1+\,|J_2|)}{3\,J_1(1+2\gamma)\sigma(n,3)+4(1+\gamma)^2\big(\sigma(n,2)\big)^2}+\frac{J_1}{3(1+2\gamma)\sigma(n,3)}.$$

■

Given (1.4) $b_2=-a_2$ and the resulting upper bound for $|a_2|$ is also true for $|b_2|$. Again, since $b_3=2a_2^2-a_3$ to obtain the upper bound for $|b_3|$ we need brief calculations which we will deal with at the next result. Therefore, at this step, we will provide upper bounds for the first two coefficients of the Taylor-Maclaurin expansion of f^{-1}.

Corollary 2.2. If $f\in\mathcal{H}_\Sigma(n,\gamma,\varphi)$, then

$$|2a_2^2-a_3| \le\frac{J_1(J_1+\,|J_2|)}{3\,J_1(1+2\gamma)\sigma(n,3)+4(1+\gamma)^2\big(\sigma(n,2)\big)^2} +\frac{J_1}{3(1+2\gamma)\sigma(n,3)}.$$

Proof. By the proof of the Theorem 2.1 we have

$$2a_2^2-a_3 =\frac{J_1^2(c_2+d_2)+J_1J_2d_1^2}{12\,J_1(1+2\gamma)\sigma(n,3)+16(1+\gamma)^2\big(\sigma(n,2)\big)^2} +\frac{J_1(d_2-c_2)}{12\,(1+2\gamma)\sigma(n,3)}.$$

Then, clearly, by using Lemma 1.3 we obtain

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$$|2a_2^2 - a_3| \leq \frac{J_1(J_1 + |J_2|)}{3J_1(1+2\gamma)\sigma(n,3) + 4(1+\gamma)^2(\sigma(n,2))^2} + \frac{J_1}{3(1+2\gamma)\sigma(n,3)}.$$

■

Considering the Taylor-Maclaurin series of e^z, a special case is obtained by taking $\varphi(z) = e^z$ in the Theorem 2.1. The details are shown below.

Corollary 2.3. Consider the function $f \in \mathcal{H}_\Sigma(n,\gamma,e^z)$ then

$$|a_2| \leq \sqrt{\frac{3}{6(1+2\gamma)\sigma(n,3) + 8(1+\gamma)^2(\sigma(n,2))^2}}$$

and

$$|a_3| \leq \frac{3}{6(1+2\gamma)\sigma(n,3) + 8(1+\gamma)^2(\sigma(n,2))^2} + \frac{1}{3(1+2\gamma)\sigma(n,3)}.$$

Proof. Let $\varphi(z) = e^z$. Since

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

so, by (1.5) $J_1 = 1$, $J_2 = \frac{1}{2}$, and by Theorem 2.1 we obtain

$$|a_2| \leq \sqrt{\frac{3}{6(1+2\gamma)\sigma(n,3) + 8(1+\gamma)^2(\sigma(n,2))^2}}$$

and

$$|a_3| \leq \frac{3}{6(1+2\gamma)\sigma(n,3) + 8(1+\gamma)^2(\sigma(n,2))^2} + \frac{1}{3(1+2\gamma)\sigma(n,3)}.$$

■

Now, if we take

$$\varphi(z) = \frac{1+(1-2\delta)z}{1-z}; \quad 0 \leq \delta < 1, \quad z \in \mathcal{D},$$

in the Theorem 2.1 then, $J_1 = J_2 = 2(1-\delta)$ and we get another result.

Corollary 2.4. Consider the function $f \in \mathcal{H}_\Sigma\left(n,\gamma,\frac{1+(1-2\delta)z}{1-z}\right)$ where, $0 \leq \delta < 1$ and $z \in \mathcal{D}$ then,

$$|a_2| \leq \sqrt{\frac{4(1-\delta)^2}{3(1-\delta)(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2}}$$

and

$$|a_3| \leq \frac{4(1-\delta)^2}{3(1-\delta)(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2} + \frac{2(1-\delta)}{3(1+2\gamma)\sigma(n,3)}.$$

Proof. We note that

$\frac{1+(1-2\delta)z}{1-z} = 1 + 2(1-\delta)z + 2(1-\delta)z^2 + \cdots, \quad 0 \leq \delta < 1, z \in \mathcal{D}.$

So, by (1.5) $J_1 = J_2 = 2(1-\delta)$, and applying Theorem 2.1

$$|a_2| \leq \sqrt{\frac{4(1-\delta)^2}{3(1-\delta)(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2}}$$

and

$$|a_3| \leq \frac{4(1-\delta)^2}{3(1-\delta)(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2} + \frac{2(1-\delta)}{3(1+2\gamma)\sigma(n,3)}.$$

■

Finally, upon letting

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha; \ 0 < \alpha \leq 1, \quad z \in \mathcal{D}$$

we obtain the following new result.

Corollary 2.5. Consider the function $f \in \mathcal{H}_\Sigma\left(n,\gamma,\left(\frac{1+z}{1-z}\right)^\alpha\right); 0 < \alpha \leq 1, z \in \mathcal{D}$ then,

$$|a_2| \leq \sqrt{\frac{2\alpha^2(1+\alpha)}{3\alpha(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2}}$$

and

$$|a_3| \leq \frac{2\alpha^2(1+\alpha)}{3\alpha(1+2\gamma)\sigma(n,3) + 2(1+\gamma)^2(\sigma(n,2))^2} + \frac{2\alpha}{3(1+2\gamma)\sigma(n,3)}.$$

Proof. Since

$\left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots,$

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then by (1.5) $J_1 = 2\alpha,\ J_2 = 2\alpha^2$ and applying Theorem 2.1

$$|a_2| \leq \sqrt{\frac{2\alpha^2(1+\alpha)}{3\alpha(1+2\gamma)\sigma(n,3)+2(1+\gamma)^2\big(\sigma(n,2)\big)^2}}$$

and

$$|a_3| \leq \frac{2\alpha^2(1+\alpha)}{3\alpha(1+2\gamma)\sigma(n,3)+2(1+\gamma)^2\big(\sigma(n,2)\big)^2} + \frac{2\alpha}{3(1+2\gamma)\sigma(n,3)}.$$

3. Conclusion:

In this article, our investigation is due to the fact that we can find interesting and useful applications of special functions and especially bi-univalent functions .The new bi-univalent function subclasses $\mathcal{H}_{\Sigma}(n,\gamma,\varphi)$ in the open disk $\mathcal{D}$, was examined in this paper. We explored the coefficients of $|a_2|$ and $|a_3|$ in the Taylor series of them. Additionally, we discovered some corollaries and implications of the primary findings. furthermore, the provided bounds improve and extend some previous results.

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References

[1] Lewin,M. (1967). On a coefficient problem for bi-univalent functions. *Proceedings of the American mathematical society*, *18*(1), pp.63-68.

[2] Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in¦ z¦< 1. *Archive for Rational Mechanics and Analysis*, *32*(2), pp.100-112.

[3] Brannan, D. A. (1980) The Lowner Differential Equation. In: *Aspects of contemporary complex analysis, (pp. 79-95)*. Academic Press INC. London (University of Durham).

[4] Srivastava, H.M., Mishra, A.K. and Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied mathematics letters*, *23*(10), pp.1188-1192.

[5] Al-Ameedee, S.A., Atshan, W.G. and Al-Maamori, F.A. (2020). May. Coefficients estimates of bi-univalent functions defined by new subclass function. In *Journal of Physics: Conference Series* (Vol. 1530, No. 1, p. 012105). IOP Publishing.

[6] Atshan, W.G., Rahman, I.A.R. and Lupaş, A.A. (2021). Some results of new subclasses for bi-univalent functions using quasi subordination. *Symmetry*, *13*(9), p.1653.

[7] Shehab, N.H. and Juma, A.R.S. (2021). Application of Quasi Subordination Associated with Generalized Sakaguchi Type Functions. *Iraqi Journal of Science*, *62*(12), pp.4885-4891.

[8] Hameed Mohammed, N. (2021). Coefficient Bounds for a New Class of Bi-Univalent Functions Associated with Subordination. *Mathematical Analysis and Convex Optimization*, *2*(2), pp.73-82.

[9] Ruscheweyh, S. (1975). New criteria for univalent functions. *Proceedings of the American Mathematical Society*, *49*(1), pp.109-115.

[10] Hussain, S.,et al. (2017). Coefficients bounds for certain subclass of biunivalent functions associated with Ruscheweyh-Differential operator. *Journal of Complex Analysis*, *2017*.

[11] Duren, P. L. (1983). "Grundlehren der mathematischen Wissenschaften."*Springer-Verlag NewYork, Berlin, Heidelberg, and Tokyo.* Vol. 259**.**

[12] Srivastava, H.M. and Owa, S. (1992). *Current topics in analytic function theory*. World Scientific.

[13] Cho, N.E., et al. (2019). The second Hankel determinant problem for a class of bi-close-to-convex functions. *Mathematics*, *7*(10), p.986.

[14] Srivastava, H.M., Gaboury, S. and Ghanim, F. (2018). Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma–Minda type. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, *112*(4), pp.1157-1168.

[15] Ma, W. (1992). A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis, 1992*. International Press Inc.

[16] Mendiratta, R., Nagpal, S. and Ravichandran, V. (2015). On a subclass of strongly starlike functions associated with exponential function. *Bulletin of the Malaysian Mathematical Sciences Society*, *38*(1), pp.365-386.

[17] ALREFAI, O. and Ali, M. (2020). General coefficient estimates for bi-univalent functions: a new approach. *Turkish Journal of Mathematics*, *44*(1), pp.240-251.