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A New Scaled Three-Term Conjugate Gradient Algorithms for Unconstrained Optimization

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A r t i c l e i n f o.

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ABSTRACT

Since optimization problems are getting more complicated, new ways to solve them must be thought of, or existing methods must be improved. In this research, we expand the different parameters of the three-term conjugate gradient method to work out unconstrained optimization problems. Our new CG approach meets the conditions of sufficient descent, and global convergence. In addition, we describe some numerical results that imply comparisons to relevant methodologies in the existing research literature.

خوارزميات متدرجة مترافقة جديدة ثلاثية الحدود في الامثلية غير المقيدة

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الملخص

منذ أن أصبحت مسائل الامثلية أكثر تعقيداً، تم التفكير في طرائق جديدة أو طرائق موجودة لحلها والتي يجب تطويرها. في هذا البحث وسّعنا بتوسيع معلمات مختلفة لطريقة التدرج المترافق من النوع الثلاثي الحدود (Three–Term) لحل مشاكل التحسين غير المقيدة. تتميز طريقة CG الجديدة المقترحة لدينا بشروط الترافق وخاصية الانحدار الكافية وخصائص تقارب الشاملة. نقوم أيضًا باختبار وادراج عن النتائج العددية التي تقدم مقارنات للطرق ذات الصلة في الدراسات السابقة.

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Introduction

The conjugate gradient method(CG)plays an important role in solving the unconstrained optimization problem .In general, the method has the following form [1]

$\min f(x)\; x\epsilon R^n$ (1.1)

Where $f: R^n \rightarrow$R is continuously differentiable the(CG)method is an iterative method of the form

$x_{u+1} = x_u + \alpha_u d_u, u = 0,1,2 \ldots$ (1.2)

Where x_u is the current iterate point, and $\alpha_u > 0$

The step size, d_u is asearch direction where it is defined as:

$$d_u = \begin{cases} -g_u & u = 0 \\ -g_{u+1} + \beta_u d_u & u \geq 1 \end{cases} \quad (1.3)$$

Where $g_u = \nabla f(x_u)$,$\beta_u \epsilon R$ conjugacy scalar parameter [2].

Some well-known formulas are given as follows [3]:

$$\beta_u^{HS} = \frac{g_{u+1}^T y_u}{d_u^T y_u} \quad (1.4a)$$

$$\beta_u^{FR} = \frac{g_{u+1}^T g_u}{g_u^T g_u} \quad (1.4b)$$

$$\beta_u^{Dy} = \frac{g_{u+1}^T g_{u+1}}{d_u^T y_u} \quad (1.4c)$$

$$\beta_u^{LS} = \frac{g_{u+1}^T y_u}{-d_u^T g_u} \quad (1.4d)$$

$$\beta_u^{CD} = \frac{\|g_{u+1}\|^2}{-d_u^T g_u} \quad (1.4e)$$

$$\beta_u^{PRP} = \frac{g_{u+1}^T y_u}{g_u^T g_u} \quad (1.4f)$$

New Method and its Algorithm:

$$d_{u+1} = -g_{u+1} + \beta_u s_u - \theta_u y_u \quad (2.1)$$

Is general three terms direction forms formulated where the parameters β=any standard conjugant parameter $(FR), (PR), (HS), (LS), (DY), (DX), and\ (BA)$ with others and θ_u can we see in many three-term conjugate gradient algorithms[4]

$$d_{u+1} = -g_{u+1} + \beta_u^{FR} d_u - \frac{d_u^T g_{u+1}}{g_u^T g_u} g_{u+1} \quad (2.2)$$

Where

$$\theta = \frac{d_u^T g_{u+1}}{g_u^T g_u} \quad (2.3)$$

$$d_{u+1} = -g_{u+1} + \beta_u^{HS} d_u - \frac{g_{u+1}^T d_u}{d_u^T y_u} y_u \quad (2.4)$$

Where

$$\theta = \frac{d_u g_{u+1}^T}{d_u^T y_u}$$

Another general TT direction is parametrizing with λ by:

$$d_{u+1} = -g_{u+1} + \beta_u^{HS} d_u - \lambda \mu y_u, \lambda > 1 \quad (2.5)$$

$$\mu = \theta = \frac{d_u g_{u+1}^T}{d_u^T y_u}$$

We are interested in the TT formulas given by (2.5) for the control parameter founded (λ), and in other to derive our methods, we start from equation (2.2), but we use DY instead to FR, so we equivalence (2.2) by (2.5), with DY

$$\beta^{DY} d_u - \frac{d_u^T g_{u+1}}{g_u^T g_u} g_{u+1} = \beta_u^{new} d_u - \lambda \theta_u g_{u+1} \quad (2.6)$$

Setting β^{DY}what equal and θ as (2.3)

$$\frac{g_{u+1}^T g_{u+1}}{d_u^T y_u} d_u - \frac{d_u^T g_{u+1}}{g_u^T g_u} g_{u+1} = \beta_u^{New} d_u - \lambda \frac{d_u^T g_{u+1}}{g_u^T g_u} g_{u+1}$$

Multiplying both sides of the above equation with direction d_u^T at $u - th$ iterate we get

$$\frac{\| g_{u+1}\|^2}{d_u^T y_u} d_u^T d_u - \frac{(d_u^T g_{u+1})(d_u^T g_{u+1})}{g_u^T g_u} = \beta_u^{New} d_u^T d_u - \lambda \frac{(d_u^T g_{u+1})(d_u^T g_{u+1})}{g_u^T g_u}$$

Were β_u^{New} as in (2.6) and θ $= \frac{d_u^T g_u}{\|g_u\|^2}$

$$\frac{\|g_{u+1}\|^2}{d_u^T y_u} - \frac{(d_u^T g_{u+1})(d_u^T g_{u+1})}{(g_u^T g_u)(d_u^T d_u)} = \beta_u^{New} - \lambda \frac{(d_u^T g_{u+1})(d_u^T g_{u+1})}{(g_u^T g_u)(d_u^T d_u)}$$

$$\beta_u^{New} = \beta_u^{DY} + (\lambda - 1)\frac{(d_u^T g_u)^2}{\|g_u\|^2 \|d_u\|^2} \quad (2.7)$$

Were || || Euclidean norm. Then according to general (2.1) and (2.5) and our derived parameter of conjugate (2.6) will be

$$d_{u+1}^{New} = -g_{u+1} - \beta_u^{New}\, d_u \qquad (2.8)$$

Where β_u^{New} as in (2.7).

3.1 The DY-TT CG-Algorithms:

Step1: Given an initial point $x_0 \in R^n$ and positive parameters, $\psi = 0.2,\ 0 \le \delta \le 0.5$ and $\delta \le \sigma \le 1$. Set the initial search direction $d_0 = -g_0$ and let $u = 0$.

Step2: If $\|g_u\| \le \varepsilon,$ then stop.

Step3: Determine step length $\alpha_u > 0$ satisfying the Strong Wolfe Condition (4) with computing $x_{u+1} = x_u + \alpha_u\, d_u$

Step4: Compute the new search direction (2.8), where the conjugacy parameterβ_u are known in (2.7).

Step5: If $|g_{u+1}^T g_u| \ge \varphi\|g_{u+1}\|^2$, then go to **Step (1)** else continue. (this is **Powell restart**)[5].

Step6: Let $u = u + 1$ and go to **Step (2)**.

3.1 The Descent Property of the New formals

We will mention the proof of the descent property of the new proposed formula, the conjugated descent property algorithm.

Theorem 3.1 (decent property):

Suppose that the step-size α_u hold the Wolfe condition. The direction of the search d_{u+1}^{New} with the parameter β_u^{New} given in equation (2.8) satisfying the descent property for all $u \ge 1$.

Proof: we began by multiplying the direction (2.8) by g_{u+1}^T

$$g_{u+1}^T d_{u+1}^{New} = -g_{u+1}^T g_{u+1} + \beta_u^{New}\, g_{u+1}^T d_u$$

Replace the conjugacy parameter with its equal to

$$g_{u+1}^T d_{u+1} = -g_{u+1}^T g_{u+1} + \left[\beta_u^{DY} + (\lambda - 1)\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2}\right] g_{u+1}^T d_u$$

$$g_{u+1}^T d_{u+1} = -g_{u+1}^T g_{u+1} + \frac{g_{u+1}^T g_{u+1}}{y_u^T d_u} g_{u+1}^T d_u + (\lambda - 1)\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2} g_{u+1}^T d_u \ (3.1)$$

if the step length α_u is chosen by an exact line search which requires

$g_{u+1}^T d_{u+1} = 0.$

then the proof is complete. If the step length α_u is chosen by inexact line search which requires $g_{u+1}^T d_{u+1} \neq 0$ the first two terms of equation (3.1) are less than or equal to zero because the parameter of (DY) satisfies the descent condition, and the third term is less than or equal to zero, because $\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2} \ge 0$ and λ<1, so,

$$g_{u+1}^T d_{u+1} \le 0$$

On some studies of the CG-methods, the sufficient descent or descent condition plays an important role, but unfortunately sometimes, this condition is hard to hold [6].

3.2 Global Convergence analysis

We will proof in this paragraph that the conjugated gradient method with three limits converge absolutely, we need the following hypotheses to study the convergence of the proposed new algorithm:

Assumption (H):

(i) The level set $S = \{x : x \in R^n, f(x) \le f(x_0)\}$ is bounded, where x_0 is the starting point, and there exists a positive constant such that, for all: $B > 0$ and defined below[7].

(ii) In a neighborhood Ω of S, f is continuously differentiable and its gradient g is Lipschitz continuously, namely, there exists a constant $L \ge 0$ such that

$$\|g(x) - g(x_u)\| \le L\|x - x_u\|, \forall x, x_u \in \Omega \qquad (3.1)$$

Obviously, from the Assumption (H, i) there exists a positive constant D such that:

$$B = max\{\|x - x_u\|, \forall x, x_u \in s\} \qquad (3.2)$$

Where B is the diameter of Ω. From Assumption (H, ii), we also know that there exists a constant $\gamma \ge 0$, such that[8]:

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$\underline{\gamma} \le \|g(x)\| \le \bar{\gamma}, \forall x \in S$ (3.3)

(iii) Suppose the function uniformly convex, there exist a constant τ

$$\left(\nabla f(x) - \nabla f(y)\right)^T (x-y) \ge \tau\|x-y\|^2 \text{for all } \in \Omega \quad (3.4)$$

The inequality above is equivalent to the inequality below[9]:

$y^T s \ge \tau\|s\|^2$ and $\tau\|s\|^2 \le y^T s \le L\|s\|^2$ (3.5)

Lemma (3.2)[7]**:** Suppose assumption (H) holds, consider the iteration process of the form (2)-(3), where d_{u+1} satisfies the descent condition ($d_u^T g_k \le 0$) for all u$\ge$ 1 and $\propto_u$ satisfies SWC. Then

$$\sum_{u\ge1} \frac{(g_u^T d_u)^2}{\|d_u\|^2} < +\infty \quad (3.6)$$

Proof: From the first inequality in SWC we can get:

$$f_{u+1} - f_u \le \sigma\alpha_u g_u^T d_u$$

Combining this with the results in reality $\left\{\alpha_u \ge \frac{(1-\sigma)|d_u{}^T g_u|}{L\|d_u\|^2}\right\}$. yields

$$f_{u+1} - f_u \le \frac{\delta(1-\sigma)}{L}\frac{(g_u{}^T d_u)^2}{\|d_u\|^2} \quad (3.7)$$

Using the bound-ness of function f in Assumption (H), hence

$$\sum_{u\ge1} \frac{(g_u^T d_u)^2}{\|d_u\|^2} < +\infty \quad (3.8)$$

Theorem (3.3)

Suppose that assumption H holds and consider the new algorithm obtained by DY which denoted (TT-SZ Algorithm) where $\propto_u$is computed by wolf Line Search, then

$$\lim_{u\to\infty} inf\|g_u\| = 0$$

Proof:

The proof well done by contradiction, so we suppose that the conclusion is not true, then $\|g_u\| \ne 0$, as mentioned above there exist a constant $\varsigma, \gamma > 0$ such that

$$0 < \varsigma \le \|g_k\| \le \gamma, \quad \forall k \ge 0$$

We have by equations (1.3) and (2.8) $d_{u+1} + g_{u+1} = \beta_{u+1}^{New} d_u$

Now by taking the square norm of both sides of our new direction

$$\|d_{u+1}\|^2 = (\beta_{u+1}^{New})^2\|d_u\|^2 - 2g_{u+1}^T d_{u+1} - \|g_{u+1}\|^2 \quad (3.9)$$

Divide the two sides of the equation (3.9) by $(g_{u+1}^T d_{u+1})^2$, therefore we end up with

$$\frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} = \frac{(\beta_{u+1}^{New})^2\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} - \frac{2}{g_{u+1}d_{u+1}} - \frac{\|g_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2}$$

$$= \frac{(\beta_{u+1}^{New})^2\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} - \left(\frac{1}{\|g_{u+1}\|^2} + \frac{\|g_{u+1}\|}{g_{u+1}^T d_{u+1}}\right)^2 + \frac{1}{\|g_{u+1}\|^2}$$

$$\le \frac{(\beta_{u+1}^{New})^2\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

We set the associative parameter equal to

$$\beta_u^{DY} + (\lambda-1)\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2}$$

$$\frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \le \frac{\left(\beta_{u+1}^{DY} + (\lambda-1)\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2}\right)^2\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

$$\frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \le \frac{(g_{u+1}^T g_{u+1})^2\|d_u\|^2}{(d_u^T y_u)^2(g_{u+1}^T d_{u+1})^2} + \frac{\left(2(\lambda-1)\frac{g_{u+1}^T g_{u+1}}{d_u^T g_u}\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2}\right)\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} + \frac{\left((\lambda-1)\frac{(d_u^T g_u)^2}{\|g_u\|^2\|d_u\|^2}\right)^2\|d_u\|^2}{(g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

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$$\Rightarrow \frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \leq \frac{(g_{u+1}^T g_{u+1})^2 \|d_u\|^2}{(d_u^T y_u)^2 (g_{u+1}^T d_{u+1})^2} + 2(\lambda - 1) \frac{g_{u+1}^T g_{u+1} (d_u^T g_u)^2 \|d_u\|^2}{d_u^T g_u \|g_u\|^2 \|d_u\|^2 (g_{u+1}^T d_{u+1})^2} + (\lambda-1)^2 \frac{(d_u^T g_u)^4 \|d_u\|^2}{\|g_u\|^4 \|d_u\|^4 (g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

We know that $d_u^T g_{u+1} \leq d_u^T y_u$ and by Wolfe condition $d_u^T g_{u+1} \geq c2 d_u^T g_u \Rightarrow c2 d_u^T g_u \leq d_u^T y_u \Rightarrow -c2 g d_u^T g_u \geq -d_u^T y_u$

This implies that $\|g_u\|^2 \geq -\frac{1}{c2} d_u^T y_u \Rightarrow -c2\|g_u\|^2 \leq d_u^T y_u$

$$\Rightarrow \frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \leq \frac{(g_{u+1}^T g_{u+1})^2 \|d_u\|^2}{-c_2 \|g_u\|^2 (g_{u+1}^T d_{u+1})^2} + 2(\lambda - 1) \frac{g_{u+1}^T g_{u+1} (d_u^T g_u)^2 \|d_u\|^2}{d_u^T g_u \|g_u\|^2 \|d_u\|^2 (g_{u+1}^T d_{u+1})^2} - (\lambda-1)^2 \frac{c_2 (d_u^T g_u)^4 \|d_u\|^2}{d_u^T y_u \|g_u\|^2 \|d_u\|^4 (g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

Since all the magnitude $c_2, (\lambda-1)^2, \|d_u\|^2, (d_u^T g_{u+1})^2, d_u^T y_u, \|g_u\|^2$, $\|d_u\|^4$ and $(d_u^T g_u)^2$ are greater than zero, then

$$\Rightarrow \frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \leq 2(\lambda - 1) \frac{g_{u+1}^T g_{u+1} (d_u^T g_u)^2 \|d_u\|^2}{d_u^T g_u \|g_u\|^2 \|d_u\|^2 (g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2} \Rightarrow \frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \leq -2(1-\lambda) \frac{g_{u+1}^T g_{u+1} (d_u^T g_u)^2 \|d_u\|^2}{d_u^T g_u \|g_u\|^2 \|d_u\|^2 (g_{u+1}^T d_{u+1})^2} + \frac{1}{\|g_{u+1}\|^2}$$

$$\Rightarrow \frac{\|d_{u+1}\|^2}{(g_{u+1}^T d_{u+1})^2} \leq \frac{1}{\|g_{u+1}\|^2}$$

When $u = 0$ the above inequality yields $\|d_1\|^2 / (d_1^T g_1)^2 \leq 1/\|g_1\|^2$

Hence for all u, we conclude that $\frac{\|d_u\|^2}{(d_u^T g_u)^2} \leq \frac{1}{\|g_u\|^2}$.

Therefore $\frac{\|d_u\|^2}{(d_u^T g_u)^2} \leq \sum_{i=0}^{u} \frac{1}{\|g_i\|^2}$ So, by (3.3)

$$\frac{\|d_u\|^2}{(d_u^T g_u)^2} \leq \sum_{i=0}^{u} \frac{1}{\bar{\gamma}^2} \Rightarrow \frac{\|d_u\|^2}{(d_u^T g_u)^2} \leq \frac{1}{\bar{\gamma}^2} \sum_{i=0}^{u} 1 \Rightarrow \frac{\|d_u\|^2}{(d_u^T g_u)^2} \leq \frac{u}{\bar{\gamma}^2}$$

after we take summation both sides

$$\Rightarrow \frac{\|d_u\|^2}{(d_u^T g_u)^2} \geq \frac{\gamma^2}{u} = \infty$$

Which contradicts Zountendijk condition in theorem (3.3) The proof is then complete.

Numerical Expermental

In this section we present the performance of FORTRAN implementation of the new algorithm (TT-SZ) derived in this papter on the same set of unconstrained optimization test problems used Andrie(2008)[10]. And the Matlab code well written to illustrate the comparsion using Dolan Molar method to figure out the strength the new techniques. These algorithm are compared with two well known Three Trem Fletcher Reeves conjugate gradient algorithm introduced by Zhang [4] which is considered as one of the best TT-CG methods(Andrei,2008b). For each algorithm we have considered numerical experiments with number of variables n=100 -1000 inceasing 100. All these algorithms are implemented with the same line search procedure. Our comparisons includes the following:

1- NOI: the number of iteration
2- NOF :number of function and gradient evaluations which are same in these algorithms.
3- CPU:The total time required to solve (15) problem in the particular

dimension.

Figures (4.1), (4.3) and (4.3) gives the NOI, NOF and total time required for solving (15) problems for n=100, 200, 300, 400, 500, 600, 700, 800, 900,

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1000 respectively. time not considered since some algorithms are failed to arrive to the solution in less than maximum number of iterations which is 2000.

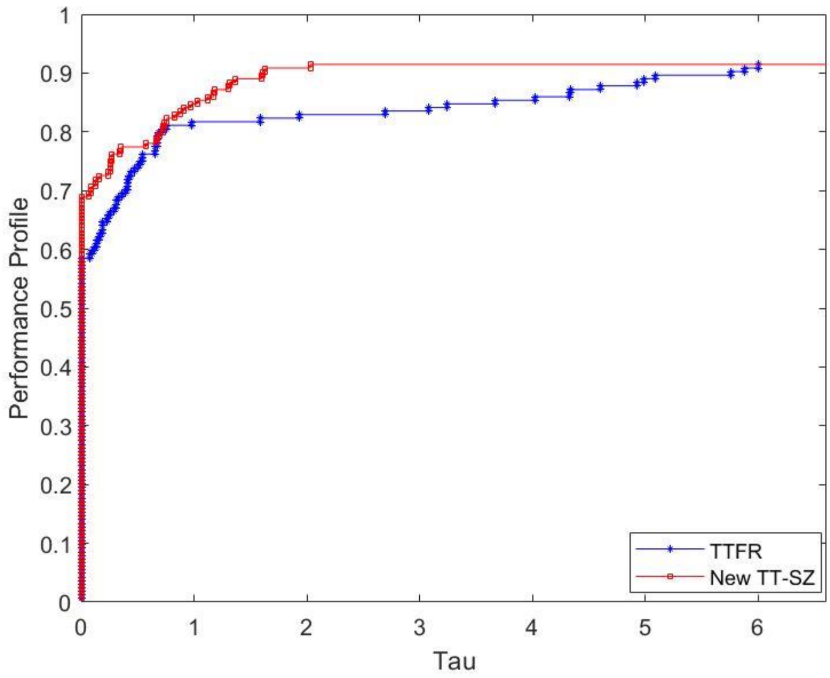

Figure (4.1): ITER comparison between TT-SZ and TT-FR.

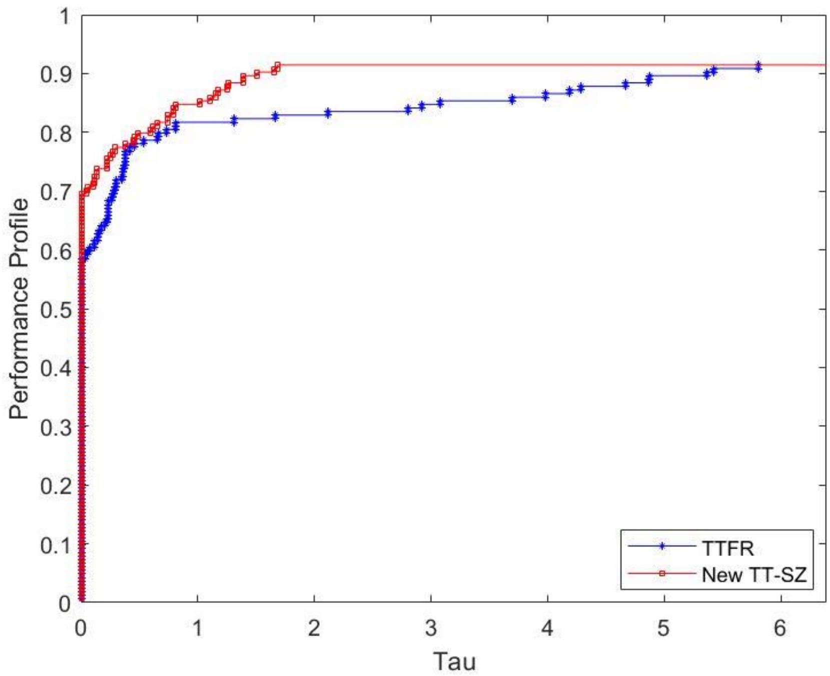

Figure (4.2): NOFG comparison between TT-SZ and TT-FR.

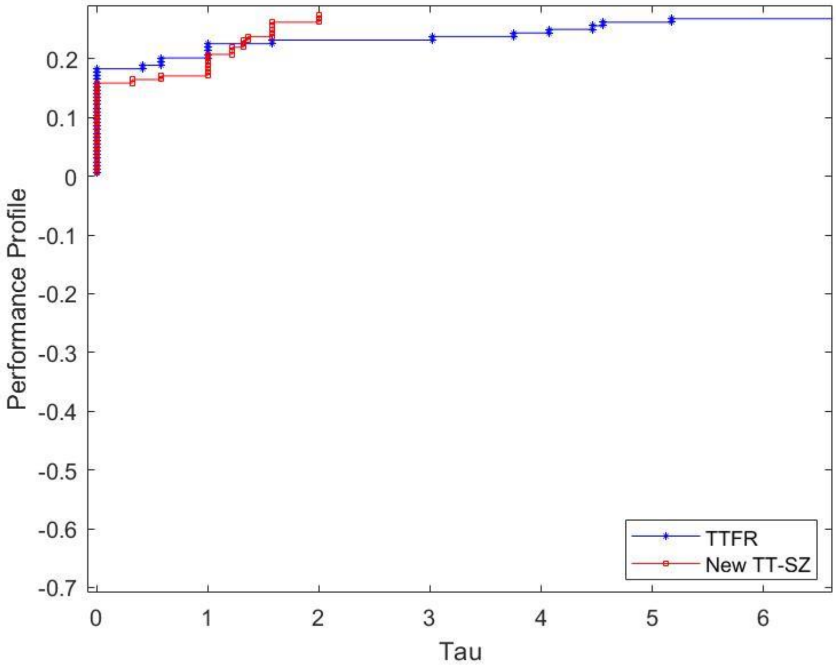

Figure (4.3): CPU comparison between TT-SZ and TT-FR.

We can see that our method's curve, which is shown in red, goes beyond the competition's curve, which is shown in blue. This is true for the approofd comparison criteria, which are the number of iterations, the number of function calculations, and the time.

Conclusion

The expansion of different parameters of the conjugated gradient method of trinomial type is investigated to solve unconstrained optimization problems. We also discussed the analytical side of the proposed algorithm and how the conjugation condition, the sufficient descent property, and the Global convergence properties were used to study its stability. As for how the research can be used in the real world, it has been checked for optimal mathematical functions and the addition of numerical results that show how similar methods from other studies compare.

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