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n-absorbing I-primary ideals in commutative rings

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A r t i c l e i n f o.

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ABSTRACT

We define a new generalization of n-absorbing ideals in commutative rings called n-absorbing I-primary ideals. We investigate some characterizations and properties of such new generalization. If P is an n-absorbing I-primary ideal of R and √IP=I√P, then √P is a n-absorbing I-primary ideal of R. And if √P is an (n-1)-absorbing ideal of R such that √(I√P) ⊆IP, then P is an n-absorbing I-primary ideal of R.

المثالي الأولي المختزل n من نوع I في الحلقات التبديلية

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الملخص

عرفنا تعميماً جديداً للمثالى المختزل n في الحلقات التبديلية والذي يسمى المثالي الأولي المختزل n من نوع I . حيث تم الاستقصاء عن بعض الميزات والخصائص لهذا التعميم الجديد. فإذا كانت P مثالي أولي مختزل لـــ R و $\sqrt{IP} = I\sqrt{P}$ ، فإن √P هو مثالي أولي مختزل n من نوع I. وإذا كانت $\sqrt{P}$ مثالي مختزل من نوع $(n-1)$ في R فإن $\sqrt{P}$ مثالي مختزل مع $\sqrt{(I\sqrt{P})} \subseteq IP$ ، وأيضاً P مثالي مختزل n من نوع I في R.

Introduction

In our article all rings are commutative ring with non-zero identity. In the recent years many generalizations of prime ideals were defined. Here state some of them. The notion of a weakly prime ideal was introduced by Anderson and Smith, where a proper ideal P of a commutative ring R is a weakly prime if $x, y \in R$, and $0 \neq xy \in P$ then $x \in P$ or $y \in P$ in [5] . Ebrahimi Atani and Farzalipour defined the nation of weakly primary ideals [8] . The authors in [6] and [3] introduced the notions $2-$ absorbing and $n-$absorbing ideals in commutative rings. A proper ideal P is said to be $2-$absorbing (or $n-$absorbing) ideal if whenever the product of three (or $n+1$) elements of R in P, the product of two (or n) of these elements is in P.

In [1] and [2], the author Akray introduced the notions $I-$prime ideal, $I-$primary ideal and $n-$absorbing $I-$ideal in commutative rings as a generalization of prime ideals. For fixed proper ideal I of a commutative ring R with identity, a proper ideal P of R is an $I-$primary if for $c, d \in R$ with $cd \in P - IP$, then $c \in P$ or $d \in \sqrt{P}$. Throughout the paper the notation $b_1 \cdots \widehat{b_i} \cdots b_n$ means that b_i is excluded from the product $b_1 \cdots b_n$. A proper ideal P of R is an $n-$absorbing $I-$primary ideal if for $b_1, \cdots, b_{n+1} \in R$ such that $b_1 \cdots b_{n+1} \in P - IP$, then $b_1 \cdots b_n \in P$ or $b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1} \in \sqrt{P}$ for some $i \in \{1,2,\cdots,n\}$ where $\sqrt{P}$ is the radical of the ideal P.

Assume that R is an integral domain with quotient field F. The authors in [7] introduced a "proper ideal P of R is a strongly primary if, whenever $cd \in P$ with $c, d \in F$, we have $c \in P$ or $d \in \sqrt{P}$. In [9], a proper ideal P of R is a strongly $I-$primary ideal if $cd \in P - IP$ with $c, d \in F$, then $c \in P$ or $d \in \sqrt{P}$. It is said that a proper ideal P of R is quotient $n-$absorbing $I-$ primary" if $b_1 b_2 \cdots b_{n+1} \in P$ with $b_1, b_2, \dots, b_{n+1} \in F$, then $b_1 b_2 \cdots b_n \in P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$. Set P is an ideal of a ring R, let P be an $n-$absorbing $I-$primary ideal of R and $b_1, \dots, b_{n+1} \in R$. The statement is that $(b_1, \dots, b_{n+1})$ is an $I-(n+1)-$ tuple of P if $b_1 \cdots b_{n+1} \in IP$, $b_1 b_2 \cdots b_n \notin P$ and for any $1 \leq i \leq n$, $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \notin \sqrt{P}$.

2 $n-$absorbing $I-$primary ideals

In this section, we start with to define the definition of an $n-$absorbing $I-$primary ideal of a ring R.

Definition: A proper ideal P *of* R *is an* $I-$*primary* if for $c, d \in R$ *with* $cd \in P - IP$*, then* $c \in P$ *or* $d \in \sqrt{P}$.

Definition: A proper ideal P of R *is an* $n-$*absorbing* $I-$*primary ideal if for* $b_1, \cdots, b_{n+1} \in R$ *such that* $b_1 \cdots b_{n+1} \in P - IP$ *, then* $b_1 \cdots b_n \in P$ *or* $b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n+1} \in \sqrt{P}$ *for some* $i \in \{1,2,\cdots,n\}$.

Example 2.1 Consider the ring $A = k[t_1, t_2, \dots, t_{n+2}]$, *where* k *is a field and suppose that* $P = \langle t_1 t_2 \cdots t_{n+1}, t_1^2 t_2 \cdots t_n, t_1^2 t_{n+2} \rangle$, $I = \langle t_1 t_2 \cdots t_n, t_1 t_2 \cdots t_{n+1} \rangle$. *Then* $P - IP = \langle t_1 \cdots t_{n+1}, t_1^2 t_2 \cdots t_n, t_1^2 t_{n+2} \rangle - \langle t_1 t_2 \cdots t_{n+1}, t_1^2 \cdots t_n, t_1^2 \cdots t_{n+1}, t_1^2 t_2 \cdots t_{n+2} \rangle$. *Hence* P *is an* $n-$*absorbing* $I-$*primary ideal but* P *is not* $n-$*absorbing.*

Proposition 2.2 *We set that* R *is a ring. Based on this, the following statements can be considered equivalent:*

(i) P is an $n-$absorbing $I-$primary ideal of R;

(ii) For any elements $\alpha_1, \dots, \alpha_n \in R$ with $\alpha_1 \cdots \alpha_n$ not in $\sqrt{P}$, $(P:_R \alpha_1 \cdots \alpha_n) \subseteq \left[\cup_{i=1}^{n-1} \left(\sqrt{P}:_R \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n\right)\right] \cup (P:_R \alpha_1 \cdots \alpha_{n-1}) \cup (IP:_R \alpha_1 \cdots \alpha_n)$.

Proof. (i) $\Rightarrow$ (ii) Set $\alpha_1, \dots, \alpha_n \in R$ such that $\alpha_1 \cdots \alpha_n \notin \sqrt{P}$. Let $r \in (P:_R \alpha_1 \cdots \alpha_n)$. So $r\alpha_1 \cdots \alpha_n \in P$. If $r\alpha_1 \cdots \alpha_n \in IP$, then $r \in (IP:_R \alpha_1 \cdots \alpha_n)$. Let $r\alpha_1 \cdots \alpha_n \notin IP$. Since $\alpha_1 \cdots \alpha_n \notin \sqrt{P}$, either $r\alpha_1 \cdots \alpha_{n-1} \in P$, that is, $r \in (P:_R \alpha_1 \cdots \alpha_{n-1})$ or for some $1 \leq i \leq n-1$ we have

$r\alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_n \in \sqrt{P}$, that is, $r \in (\sqrt{P} :_R \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_n)$ or $\alpha_1\alpha_2 \cdots \alpha_n \in \sqrt{P}$. By assumption the last case is not hold. Consequently $(P :_R \alpha_1 \cdots \alpha_n) \subseteq [\cup_{i=1}^{n-1} (\sqrt{P} :_R \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n)] \cup (P :_R \alpha_1 \cdots \alpha_{n-1}) \cup (IP :_R \alpha_1 \cdots \alpha_n)$

(ii) $\Rightarrow$ (i) Let $\beta_1\beta_2 \cdots \beta_{n+1} \in P - IP$ for some $\beta_1, \beta_2, \dots, \beta_{n+1} \in R$ such that $\beta_1\beta_2 \cdots \beta_n \notin P$. Then $\beta_1 \in (P :_R \beta_2 \cdots \beta_{n+1})$. If $\beta_2 \cdots \beta_{n+1} \in \sqrt{P}$, then we are done. Hence we may set $\beta_2 \cdots \beta_{n+1} \notin \sqrt{P}$ and so by part (ii), $(P :_R \beta_2 \cdots \beta_{n+1}) \subseteq [\cup_{i=2}^{n} (\sqrt{P} :_R \beta_2 \cdots \hat{\beta}_i \cdots \beta_{n+1})] \cup (P :_R \beta_2 \cdots \beta_n) \cup (IP :_R \beta_2 \cdots \beta_{n+1})$. Since $\beta_1\beta_2 \cdots \beta_{n+1} \notin IP$ and $\beta_1\beta_2 \cdots \beta_n \notin P$, the only possibility is that $\beta_1 \in \cup_{i=2}^{n} (\sqrt{P} :_R \beta_2 \cdots \widehat{\beta_i} \cdots \beta_{n+1})$. Then $\beta_1\beta_2 \cdots \widehat{\beta_i} \cdots \beta_{n+1} \in \sqrt{P}$ for some $2 \le i \le n$. Hence P is an $n-$absorbing $I-$primary ideal of R. □

Proposition 2.3 *If V be a valuation domain with the quotient field F. Then all $n-$absorbing $I-$primary ideal of V is a quotient $n-$absorbing $I-$primary ideal of R.*

Proof. We can certainly assume that P is $n-$ absorbing $I-$ primary ideal of V, and $a_1a_2 \cdots a_{n+1} \in P$ for some $a_1, a_2, \dots, a_{n+1} \in V$ such that $a_1a_2 \cdots a_n \notin P$. If $a_{n+1} \notin V$, then $a_{n+1}^{-1} \in V$. So $a_1 \cdots a_n a_{n+1} a_{n+1}^{-1} = a_1 \cdots a_n \in P$, which is a contradiction. So $a_{n+1} \in V$. If $a_i \in V$ for all $1 \le i \le n$, then there is nothing to prove. If $a_i \notin V$ for some $1 \le i \le n$, then $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in P \subseteq \sqrt{P}$. Consequently, P is a quotient $n-$ absorbing $I-$primary. □

Proposition 2.4 *Set P be an $n-$ absorbing $I-$primary ideal of R such that $\sqrt{IP} = I\sqrt{P}$, then $\sqrt{P}$ is a $n-$absorbing $I-$primary ideal of R.*

Proof. Let us assume $a_1a_2 \cdots a_{n+1} \in \sqrt{P} - I\sqrt{P}$ for some $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{P}$ for every $1 \le i \le n$. Thus, we have $n \in \mathbb{N}$ such that $a_1^n a_2^n \cdots a_{n+1}^n \in P$. If $a_1^n a_2^n \cdots a_{n+1}^n \in IP$, then $a_1a_2 \cdots a_{n+1} \in \sqrt{IP} = I\sqrt{P}$, which is a contradiction. Since P is an $n-$ absorbing $I-$ primary, our hypothesis implies $a_1^n a_2^n \cdots a_n^n \in P$. So $a_1a_2 \cdots a_n \in \sqrt{P}$ and $\sqrt{P}$ is an $n-$absorbing $I-$primary ideal of R.

Theorem 2.5 *Assume that "for any $1 \le i \le k$, I_i is an n_i-absorbing $I-$primary ideal of R such that $\sqrt{P_i} = q_i$ is an n_i- absorbing $I-$ primary ideal of R, respectively. Let $n = n_1 + n_2 + \cdots + n_k$. The following statements does hold:*

(1) $P_1 \cap P_2 \cap \cdots \cap P_k$ is an $n-$absorbing $I-$primary ideal of R.

(2) $P_1P_2 \cdots P_k$ is an $n-$absorbing $I-$primary ideal of R".

Proof. The proof of the two parts is similar, so we prove just the first. Let $H = P_1 \cap P_2 \cap \cdots \cap P_k$. Then $\sqrt{H} = P_1 \cap P_2 \cap \cdots \cap P_k$. Let $a_1a_2 \cdots a_{n+1} \in H - IH$ for some $a_1, a_2, \dots, a_{n+1} \in R$ and $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{H}$ for any $1 \le i \le n$. By, $\sqrt{H} = P_1 \cap P_2 \cap \cdots \cap P_k$ is an $n-$absorbing $I-$primary, then $a_1a_2 \cdots a_n \in P_1 \cap P_2 \cap \cdots \cap P_k$. We prove that $a_1a_2 \cdots a_n \in H$. For all $1 \le i \le k, P_i$ is an n_i-absorbing $I-$primary and $a_1a_2 \cdots a_n \in P_i - IP_i$, then we have $1 \le \beta_1^i, \beta_2^i, \dots, \beta_{n_i}^i \le n$ such that $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in P_i$. If $\beta_r^l = \beta_s^m$ it is for two couples l, r and m, s, then

$$a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} \cdots a_{\beta_1^l} a_{\beta_2^l} \cdots a_{\beta_r^l} \cdots a_{\beta_{n_l}^t} \cdots a_{\beta_1^m} a_{\beta_2^m} \cdots \widehat{a_{\beta_m^m}} \cdots a_{\beta_{m_m}^m} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} \in \sqrt{H}$$

Therefore $a_1 \cdots \widehat{a_{\beta_n^m}} \cdots a_n a_{n+1} \in \sqrt{H}$, which is a contradiction. So β_j^i 's is distinct. Hence $\left\{a_{\beta_1^1}, a_{\beta_2^1}, \dots, a_{\beta_{n_1}}, a_{\beta_1^2}, a_{\beta_2^2}, \dots, a_{\beta_{n_2}^2}, \dots, a_{\beta_1^k}, a_{\beta_2^k}, \dots, a_{\beta_{n_k}^k}\right\} = \{a_1, a_2, \dots, a_n\}$. If $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in P_i$ for any $1 \le i \le k$, then

$$a_1a_2 \cdots a_n = a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} \in H,$$

thus, we are done. Therefore, we may assume that $a_{\beta_1^1}a_{\beta_2^1}\cdots a_{\beta_{n_1}^1}\notin P_1$. Since P_1 is $I=n_1-$absorbing $I-$primary and

$$a_{\beta_1^1}a_{\beta_2^1}\cdots a_{\beta_{n_1}}a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}$$
$$=a_1\cdots a_{n+1}\in P_1-IP_1,$$

then we have $a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in P_1$.

On the other hand,

$a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in P_2\cap\cdots\cap P_k$.

Consequently $a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}a_{n+1}\in\sqrt{H}$, which is a contradiction. Similarly, $a_{\beta_1^i}a_{\beta_2^i}\cdots a_{\beta_{n_i}^i}\in P_i$ for every $2\le i\le k$. Then $a_1a_2\cdots a_n\in H$.

Proposition 2.6 *Assume that P "is an ideal of a ring R with $\sqrt{I\sqrt{P}}\subseteq IP$. If $\sqrt{P}$ is an $(n-1)-$absorbing ideal of R, then P is an $n-$absorbing $I-$primary ideal of R."*

Proof. Let $\sqrt{P}$ be an $(n-1)-$ absorbing, and consider $b_1b_2\cdots b_{n+1}\in P-IP$ for some $b_1,b_2,\dots,b_{n+1}\in R$ and $b_1b_2\cdots b_n\notin P$. Since

$$(b_1b_{n+1})(b_2b_{n+1})\cdots(b_nb_{n+1})=$$
$$(b_1b_2\cdots b_n)b_{n+1}^n\in P\subseteq\sqrt{P}-I\sqrt{P}.$$

Then for some $1\le i\le n$,

$$(b_1b_{n+1})\cdots\widehat{(x_ib_{n+1})}\cdots(b_nb_{n+1})=$$
$$(b_1\cdots\widehat{b_i}\cdots b_n)b_{n+1}^{n-1}\in\sqrt{P},$$

and so $b_1\cdots\widehat{b_i}\cdots b_nb_{n+1}\in\sqrt{P}$. Consequently P is an $n-$absorbing $I-$primary ideal of R.

We recall that a proper ideal Q *of* R is an $\boldsymbol{n}-$absorbing primary if $a_1,a_2,\cdots,a_{n+1}\in R$ and $a_1a_2\cdots a_{n+1}\in Q$, then $a_1a_2\cdots a_n\in Q$ or the product of a_{n+1} with $(n-1)$ of $a_1,a_2,\cdots,a_n$ is in $\sqrt{Q}$. It is clearly every *$n-$absorbing primary is an $n-$absorbing* I *$-$primary.*

Proposition 2.7 *Suppose that R is a ring and $r\in R$, a nonunit and $m\ge 2$ is not negative integer. Let $(0:_R r)\subseteq\langle a\rangle$, then $\langle r\rangle$ is an $n-$ absorbing $I-$primary, for some I with $IP\subseteq I^m$ if and only if $\langle a\rangle$ is $n-$absorbing primary.*

Proof. Let $\langle r\rangle$ be $n-$absorbing I^m-primary, and $a_1a_2\cdots a_{n+1}\in\langle r\rangle$ for some $a_1,a_2,\dots,a_{n+1}\in R$. If $a_1a_2\cdots a_{n+1}\notin\langle r^m\rangle$, then $a_1a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\hat{a}_i\cdots a_{n+1}\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. Based on this assumption, $a_1a_2\cdots a_{n+1}\in\langle r^m\rangle$. Hence $a_1a_2\cdots a_n(a_{n+1}+r)\in\langle r\rangle$. If $a_1a_2\cdots a_n(a_{n+1}+r)\notin\langle r^m\rangle$, then $a_1a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\hat{a}_i\cdots a_n(a_{n+1}+r)\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. So $a_1a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. Hence, suppose that $a_1a_2\cdots a_n(a_{n+1}+r)\in\langle r^m\rangle$. Thus $a_1a_2\cdots a_{n+1}\in\langle r^m\rangle$ implies that $a_1a_2\cdots a_nr\in\langle r^m\rangle$. Therefore, there exists $s\in R$ such that $a_1a_2\cdots a_n-sr^{m-1}\in(0:_R r)\subseteq\langle r\rangle$. Consequently $a_1a_2\cdots a_n\in\langle r\rangle$.

Proposition 2.8 *Assume V is a valuation domain and $n\in\mathbb{N}$. Let P be an ideal of V such that P^{n+1} is not principal. Then P is an $n-$absorbing $I^{n+1}-$primary if and only if it is an $n-$absorbing primary.*

Proof. $(\Rightarrow)$ Let P be an $n-$absorbing I^n-primary that is n't $n-$absorbing primary. Therefore, there are $a_1,\dots,a_{n+1}\in R$ such that $a_1\cdots a_{n+1}\in P$, but neither $a_1\cdots a_n\in P$ nor $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{P}$ for any $1\le i\le n$. Hence $\langle a_i\rangle\nsubseteq P$ for any $1\le i\le n+1$. And so V is a valuation domain, thus $P\subset\langle a_i\rangle$ for any $1\le i\le n+1$, and so $P^{n+1}\subseteq\langle a_1\cdots a_{n+1}\rangle$. Therefore P^{n+1} is not principal, then $a_1\cdots a_{n+1}\in P-P^{n+1}$. Therefore P is an $n-$ absorbing $I^{n+1}-$ primary implies that either $a_1\cdots a_n\in P$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{P}$ for some $1\le i\le n$, which is a contradiction. Hence P is an $n-$absorbing primary ideal of R.

$(\Leftarrow)$ Is trivial.

Theorem 2.9 We consider that $J\subseteq P$ are a proper ideal of a ring R.

(1) Let P is an $n-$absorbing $I-$primary ideal of R, then P/J is a $n-$absorbing $I-$primary ideal of R/J.

(2) Let $J \subseteq IP$ and P/J be an $n-$ absorbing $I-$primary ideal of R/J, then P is an $n-$absorbing $I-$primary ideal of R.

(3) Let $IP \subseteq J$ and P be an $n-$absorbing $I-$primary ideal of R, then P/J is a weakly $n-$absorbing primary ideal of R/J.

(4) Let $JP \subseteq IP$, J be an $n-$absorbing $I-$primary ideal of R and P/J be a weakly $n-$absorbing primary ideal of R/J, then P is an $n-$absorbing $I-$primary ideal of R.

Proof. (1) Set $b_1, b_2, \dots, b_{n+1} \in R$ such that $(b_1+J)(b_2+J)\cdots(b_{n+1}+J) \in (P/J) - I(P/J) = (P/J) - (I(P)+J)/J$. Then $b_1 b_2 \cdots b_{n+1} \in P - IP$ and from being P is an $n-$absorbing $I-$primary, we obtain $b_1 \cdots b_n \in P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{P}$ for some " $1 \leq i \leq n$. And so $(b_1+J)\cdots(b_n+J) \in P/J$ or $(b_1+J)\cdots(\widehat{b_i+J})\cdots(b_{n+1}+J) \in \sqrt{P}/J = \sqrt{P/J}$ for some $1 \leq i \leq n$. Hence we prove that P/J is $n-$absorbing $I-$primary ideal of R/J."

(2) Set $b_1 b_2 \cdots b_{n+1} \in P - IP$ for some $b_1, b_2, \dots, b_{n+1} \in R$. Then $(b_1+J)(b_2+J)\cdots(b_{n+1}+J) \in (P/J) - (I(P)/J) = (P/J) - I(P/J)$. From being P/J is an $n-$absorbing $I-$primary, we obtain that $(b_1+J)\cdots(b_n+J) \in P/J$ or $(b_1+J)\cdots(\widehat{b_i+J})\cdots(b_{n+1}+J) \in \sqrt{P/J} = \sqrt{P}/J$ for some $1 \leq i \leq n$. Therefore $b_1 \cdots b_n \in P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$, hence P is an $n-$absorbing $I-$primary ideal of R.

(3) Resulted directly from part (1).

(4) Set $b_1 \cdots b_{n+1} \in P - IP$ where $b_1, \dots, b_{n+1} \in R$. Note that $b_1 \cdots b_{n+1} \notin JP$ because $JP \subseteq IP$. If $b_1 \cdots b_{n+1} \in J$, then either $b_1 \cdots b_n \in J \subseteq P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{J} \subseteq \sqrt{P}$ for some $1 \leq i \leq n$, since J is an $n-$absorbing $I-$primary. If $b_1 \cdots b_{n+1} \notin J$, then $(b_1+J)\cdots(b_{n+1}+J) \in (P/J) - \{0\}$ and so either $(b_1+J)\cdots(b_n+J) \in P/J$ or $(b_1+J)\cdots(\widehat{b_i+J})\cdots(b_{n+1}+J) \in \sqrt{P/J} = \sqrt{P}/J$ for some $1 \leq i \leq n$. Therefore $b_1 \cdots b_n \in P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{P}$ for some $1 \leq i \leq n$. Hence P is an $n-$absorbing $I-$primary ideal of R.

Proposition 2.10 *Suppose that P is an ideal of a ring R such that IP is an $n-$absorbing primary ideal of R. If P is an $n-$absorbing $I-$primary ideal of R, then P is an $n-$absorbing primary ideal of R.*

Proof. Let $a_1 a_2 \cdots a_{n+1} \in P$ for some elements $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 a_2 \cdots a_n \notin P$. If $a_1 a_2 \cdots a_{n+1} \in IP$, then IP $n-$absorbing primary and $a_1 a_2 \cdots a_n \notin IP$ implies that $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{IP} \subseteq \sqrt{P}$ for some $1 \leq i \leq n$, and so we are done. When $a_1 a_2 \cdots a_{n+1} \notin IP$ clearly the result follows.

Theorem 2.11 *If P is an $n-$absorbing $I-$primary ideal of a ring R and $(a_1, \dots, a_{n+1})$ is an $I-(n+1)-$tuple of P for some $a_1, \dots, a_{n+1} \in R$. Then for every element $\alpha_1, \alpha_2, \dots, \alpha_m \in \{1,2,\dots,n+1\}$ which $1 \leq m \leq n$,*

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m \subseteq IP$$

Proof. We claim that by using induction on m. We take $m=1$ and assume $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} x \notin IP$ for some $x \in P$. Then $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}(a_{\alpha_1}+x) \notin IP$. Since P is a $n-$absorbing $I-$primary ideal of R and $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} \notin P$, we conclude that $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1}(a_{\alpha_1}+x) \in \sqrt{P}$, for some $1 \leq \alpha_2 \leq n+1$ different from α_1. Hence $a_1 \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1} \in \sqrt{P}$, a contradiction. Thus $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} P \subseteq IP$. Here assume that $m > 1$ and for every integer less than m the prove does hold. Let $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} x_1 x_2 \cdots x_m \notin IP$ for some $x_1, x_2, \dots, x_m \in P$. According to the induction assumption, we conclude that there exists $\zeta \in IP$ such that

$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1}+x_1)(a_{\alpha_2}+x_2)\cdots(a_{\alpha_m}+x_m)$
$= \zeta + a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}x_1x_2\cdots x_m \notin IP$

Now, we have two cases.

Case 1. Set $\alpha_m < n+1$. Since from being P is an n −absorbing I −primary, then

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_n(a_{\alpha_1}+x_1)(a_{\alpha_2}+x_2)\cdots(a_{\alpha_m}+x_m) \in P,$$

or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots \widehat{a_j} \cdots a_{n+1}(a_{\alpha_1}+x_1)(a_{\alpha_2}+x_2)\cdots(a_{\alpha_m}+x_m) \in \sqrt{P}$$

for some $j < n+1$ distinct from α_i 's; or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1}+x_1)\cdots(\widehat{a_{\alpha_i}+x_i})\cdots(a_{\alpha_m}+x_m) \in \sqrt{P}$$

for some $1 \le i \le m$. Thus either $a_1a_2\cdots a_n \in P$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{P}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{P}$, which each of these cases which is a contradiction.

Case 2. Set $\alpha_m = n+1$. Since from being P is an n −absorbing, I −primary, then

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1}+x_1)(a_{\alpha_2}+x_2)\cdots(a_{\alpha_m}+x_m) \in P,$$ or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_j} \cdots \widehat{a_{n+1}}(a_{\alpha_1}+x_1)(a_{\alpha_2}+x_2)\cdots(a_{\alpha_m}+x_m) \in \sqrt{P},$$

for some $j < n+1$ different from α_i 's; or

$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1}+x_1)\cdots(\widehat{a_{\alpha_i}+x_i})\cdots(a_{\alpha_m}+x_m) \in \sqrt{P}$ for some $1 \le i \le m-1$. Thus either $a_1a_2\cdots a_n \in P$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{P}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{P}$, which each of these cases which are a contradiction. Thus

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}I^m \subseteq IP$$

Theorem 2.12 *If P isan $n-$absorbing $I-$primary ideal of R which is not an n-absorbing primary ideal. Then*

(i) $P^{n+1} \subseteq IP$.

(ii) $\sqrt{P} = \sqrt{IP}$.

Proof. (i) Since P is assumed not to be an $n-$absorbing primary ideal of R, so P has an $I-(n+1)-$ tuple zero $(b_1,\dots,b_{n+1})$ for some $b_1,\dots,b_{n+1} \in R$. Let $c_1c_2\cdots c_{n+1} \notin IP$ for some $c_1,c_2,\dots,c_{n+1} \in P$. Therefore, according to the Theorem 2.11, there is $\lambda \in IP$ such that $(b_1+c_1)\cdots(b_{n+1}+c_{n+1}) = \lambda + c_1c_2\cdots c_{n+1} \notin IP$. Hence either $(b_1+c_1)\cdots(b_n+c_n) \in P$ or $(b_1+c_1)\cdots(\widehat{b_i+c_i})\cdots(b_{n+1}+c_{n+1}) \in \sqrt{P}$ for some $1 \le i \le n$. Thus either $b_1\cdots b_n \in P$ or $b_1 \cdots \widehat{b_i} \cdots b_{n+1} \in \sqrt{P}$ for some $1 \le i \le n$, which is a contradiction. Hence $P^{n+1} \subseteq IP$.

(ii) Clearly, $\sqrt{IP} \subseteq \sqrt{P}$. As $P^{n+1} \subseteq IP$, we obtain $\sqrt{P} \subseteq \sqrt{IP}$, we are done.

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