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Quadratic BH-algebras and Quadratic TM-algebras

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ABSTRACT
In this paper we present the notion of “quadratic” BH−algebra and TM-algebra, and show that every quadratic BH−algebra and TM-algebra $(X;*,e)$, $e \in X$ has a product of the form $x*y = x-y+e$, where $x,y \in X$; if X is a field with $|X| \geq 3$ and BCI-algebra. Moreover, we show some theorems and examples with relationships among B-algebra, BH –algebra, BG-algebra and TM-algebra.

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الجبر BH التربيعية والجبر TM التربيعية

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الملخص

في هذا البحث، نقدم مفهوم الجبر BH التربيعية و الجبر TM التربيعية, و نبين أن كل تربيعي لجبر BH التربيعية و الجبر TM التربيعية , $X \in e$, $(X;*,e)$ يكون علي الشـــكل $x * y = x - y + e$, حيث $x, y \in X$. عندما يكون X حقل $|X| \geq 3$ و يكون الجبر BCI علاوة على ذاك نعرض بعض النظريات و الأمثلة مع العلاقات بين بعض أنواع الجبر.

1. Introduction

Y. Imai and K. Iseki have presented two classes of unique algebras: BCK-algebras and BCI-algebras [1, 2]. It is acknowledged that a suitable subclass of the BCI-algebra course may be the course of BCK-algebras. Hu and Li offered a comprehensive lecture on theoretical algebras called BCH-algebras in [3, 4]. They have demonstrated that the BCI-algebra course might be a valid subclass of the BCH-algebra course. After Literatures introduced the idea of BG-algebras in 2008 [5]. The idea of BH-algebras Definition 2.5 has been introduced in another paper [6]. K. Megali and Dr. K. A. Tamilarasi [7] introduced the idea of BH-algebras in 2010. introduced the idea of B-algebras in 2022 [8]. In addition, we introduce the notion of "quadratic" BH/TM-algebras, where each quadratic BH-algebra and TM-algebra has the form $x * y = x - y + e$ where $x, y \in X$; if X is a field with $|X| \geq 3$ and BCI-algebra.

This study provides a new notion of “quadratic” BH−algebra and TM-algebra, and show that every quadratic BH−algebra and TM-algebra, $(X;*,e)$, $e \in X$ has a product of the form $x * y = x - y + e$, where $x, y \in X$; when X is a field with $|X| \geq 3$ and is BCI-algebra. Moreover, we show some theorems and examples with relationships among B-algebra, BH − algebra, BG-algebra and TM-algebra.

2. Preliminaries

Definition 2.1 An algebra (X is not an empty set; $*$ is a binary operation, e is a constant) is called a B-algebra if three requirements are satisfied:

a) $(x * x) = e$,
b) $(x * e) = x$,
c) $((x * y) * z = x * (z * (e * y)))$ (For all x, y and $z \in X$).

Example 2.2 [3] suppose that F is set of all real number except for a negative integer k, $(F,*)$ define by:

$$x * y = \frac{k(x - y)}{(k + y)} .$$

, then $(F;*,0)$ is a B-algebra.

Definition 2.3 An algebra $(X; *, 0)$ (X is not an empty set; $*$ is a binary operation ,0 is a constant) is called a BG-Algebra if three requirements are satisfied:

(1) $(x * x = 0)$,
(2) $(x * 0 = x)$,
(3) $((x * y) * (0 * y) = x)$ (For all x and y are elements of X).

Example 2.4 [5] Suppose that $(X ; * ,0)$, Where $X = \{0,1,2\}$

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

, Then $(X ; * ,0)$ is a BG-algebra.

Definition 2.5 An algebra (X is not an empty set; $*$ is binary operation ,0 is constant) is called a BH-algebra [7] if three requirements are satisfied:

a) $(x * x = 0)$,
b) $(x * 0 = x)$,
c) $((x * y) = 0$ and $(y * x) = 0 \rightarrow x = y)$ (For all x and y are elements of X).

Example 2.6 [7] Suppose that $(X;*,0)$, where X= {0, 1, 2, 3}:

*	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Therefore, $(X;* ,0)$is BH-algebra.

Definition 2.7 An algebra (X is not a null set; $*$ is binary operation, 0 is constant) is called TM-algebra if three requirements are satisfied:

i. $(x * 0 = x)$,

ii. $((x * y) * (x * z) = (z * y))$ (For all x, y *and* $z \in X$).

Example 2.8 [11] suppose that $\mathbb{Z}$ be the set of all integers, and suppose that $kx = \{kx : k\epsilon\mathbb{Z}\}$. Then$(\mathbb{Z};*,-)$ and $(k\mathbb{Z};*,-)$, are TM-algebras "$-$" is subtraction.

Proposition 2.9 [11] Let $(X;*,0)$ is a TM-algebra, then

1) $(x * x = 0)$,

2) $((x * y) * x = (0 * y))$,

3) $(x * (x * y) = y)$ (For all x *and* $y \in X$).

3. Quadratic BH-algebras and Quadratic TM-algebras

3.1 Quadratic BH-algebras

Assume X is a field with $|X| \geq 3$, an algebra $(X;*)$ is called quadratic if $x * y$ defined by

$x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$,

where $a_1, \dots a_6 \in X$ are fixed.

A quadratic algebra $(X;*)$ is said to be a quadratic BH – Algebra if for some $e \in X$ it satisfies the condition (a), (b) and (c).

Theorem 3.1.1 Assume X is a field with $|X| \geq 3$, then every quadratic BH-algebra $(X;*,e)$ has the form $x * y = x - y + e$, where $x, y \in X$.

Proof:

$(x * y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$ (3.1)

Consider (a).

$(x * x) = e = [(D + E)x + F + (A + B + C)x^2]$. (3.2)

Suppose that $x = 0$ in (3.2). Then we obtain $F = e$. Hence (3.1) it appears to be

$x * y = Ax^2 + Bxy + Cy^2 - (-Dx - Ey) + e.$ (3.3)

If $y := x$ *in* (3.1), then $e = (x * x) = e + (A + B + C)x^2 + (D + E)x,$ (3.4)

For all $x \in X$, and hence we obtain$[A + B + C] = 0 = D + E$, that is, $E = -D$ and $B = -A - C$. Hence (3.3) apparently, it is $(x * y) = (x - y)(Ax - Cy + D) + e.$ (3.5) *suppose that* $y := e$ *in* (3.5). Then by (b)

We have $x = (x * e) = e + (x - e)(Ax - Ce + D),$ (3.6)

On BH-Algebra that is, $(Ax - Ce + D - 1)(x - e) = 0.$

Because X is a field, either $(x - e) = 0$ *or* $[Ax - Ce + D] - 1 = 0.$

because $|X| \geq 3$, we have $[Ax - Ce + D] - 1 = 0$, for all $x = e$ in X. So $A = 0, 1 - D + Ce = 0.$ thus (3.5) apparently, it is

$[(x - y) + C(x - y)(e - y)] + e = x * y.$ (3.7)

To satisfy condition (c), suppose that $x * y$ and $y * x$.

$$y * x = (y - x) + C(y - x)(e - x) + e. \quad (3.8)$$

$$0 = (x * y) - (y * x) = (x - y) + C(x - y)(e - y) + e - (y - x) - C(y - x)(e - x) - e.$$

$$0 = 2(x - y) + 2C(x - y)(2e - y - x)$$

We obtain $C = 0$, then $x = y$. This means that every quadratic BH-algebra $(X;*, e)$ has of the form $x * y = x - y + e$, where $x, y \in X$, completing the proof.

Theorem 3.1.2 Assume X is a field with $|X| \geq 3$. Then every quadratic BH-algebra on X is a BCI-algebra.

Proof: It is direct from Theorem (3.1.1) and

3.2 Quadratic TM-algebras

Assume X is a field with $|X| \geq 3$, an algebra $(X;*)$ is called quadratic if $x * y$ defined by

$x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$,

where $a_1, \dots a_6 \in X$ are fixed.

A quadratic algebra $(X;*)$ is said to be a quadratic TM – Algebra if for some $e \in X$ it satisfies the condition i and ii.

Theorem 3.2.1 Assume X is a field with $|X| \geq 3$, then every quadratic TM-algebra $(X,*,e)$ has of the form $x * y = x - y + e$, where $x, y \in X$.

Proof:

$(x * y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ (3.9)

$(x * x) = e = [(A + B + C)x^2 + (D + E)x] + F$. (By Proposition 2.9) (3.10)

Let $x = 0$ in (3.10). Then we obtain $F = e$. Hence (3.9) it appears to be

$(x * y) = Ax^2 + Bxy + Cy^2 + (Dx + Ey) + e$. (3.11)

If $y = x$ in (3.11), then $e = (x * x) = -(-D - E)x + (A + B + C)x^2$ (3.12)

For any $x \in X$, and hence we consider $(A + B + C) = 0 = (D + E)$, that is, $E = -D$ and $B = -A - C$. Hence (1.3) Apparently, it is $(x * y) = [(x - y)(Ax - Cy + D)] + e$. (3.12) suppose that $y = e$ in (3.12). Then by (ii)

We have $x = x * e = [(x - e)(Ax - Ce + D)] + e$, (3.14)

On TM-Algebra that is, $[(Ax - Ce + D - 1)(x - e)] = 0$.

Because X is a field, either $(x - e) = 0$ *or* $Ax - Ce + D - 1 = 0$.

because $|X| \geq 3$, we have $(Ax - Ce + D) - 1 = 0$, for all $x = e$ in X. So $A = 0, 1 - D + Ce = 0$. Thus (3.12) apparently, it is

$[(x - y) + C(x - y)(e - y)] + e = (x * y)$. (3.15)

To satisfy condition (ii) we consider:

$[(x * y) * (x * z)] = [[(x * y) - (x * z))] + C[(x * y) - (x * z)][e - (x * z)]] + e$ (3.16)

$= [(x - y) + C(x - y)(e - y) + e - (x - z) - C(x - z)(e - z) - e] + C[(x - y) + C(x - y)(e - y) + e - (x - z) - C(x - z)(e - z) - e][e - (x - z) - C(x - z)(e - z) - e] + e$

$= [-y + z + C(x - y)(e - y) - C(x - z)(e - z)] + C[(x - y) + C(x - y)(e - y) - (x - z) - C(x - z)(e - z)][-(x - z) - C(x - z)(e - z)] + e$

$= -y + z + C(x - y)(e - y) - C(x - z)(e - z) - C(x - y)(x - z) - 2C(x - y)(e - y)(x - z) + C(x - z)2 + C2(x - z)2(e - z) - C2(x - y)(x - z)(e - z) - C3(x - y)(e - y)(x - z)(e - z) + C2(x - z)2(e - z) + C3(x - z)2(e - z)2]$

$= -y + z + C(x - y)(e - y) - C2(x - y)(x - z)(2e - y - z) + 2C2(x - z)2(e - z) - C(x - z)(e - y) + C3(x - z)(e - z)[-(x - y)(e - y) + (x - z)(e - z)]$

$= -y + z + C(e - y)(z - y) + C2(x - z)[-(x - y)(2e - y - z) + 2(x - z)(e - z)] + C3(x - z)(e - z)][-(x - y)(e - y) + (x - z)(e - z)]$

(3.17)

$z * y = z - y + C(z - y)(e - y) + e$

(3.18)

By (3.17) and (3.18) we obtain

$[(x * y) * (x * z)] - (z * y) = 0$

$= [C^2(x - z)[-(x - y)(2e - y - z) + 2(x - z)(e - z)] + C^3(x - z)(e - z)][-(x - y)(e - y) + (x - z)(e - z)]]$

Consider $C = 0$. This means that every quadratic TM-algebra $(X; *, e)$ has of the form $x * y = x - y + e$, where $x, y \in X$, completing the proof.

Examples 3.2.2 Let R be the set of all real numbers. Define $x * y := (x - y) + \sqrt{5}$, then $(R; *, \sqrt{5})$ is quadratic TM-algebra and quadratic BH-algebra.

Theorem 3.2.3 Assume X is a field with $|X| \geq 3$, then

$(X; *, 0)$ is BCI-algebra. ⟵ $(X; *, 0)$ is quadratic B-algebra. $(X; *, 0)$ is quadratic BH-algebra.

Theorem 3.2.4 Assume X is a field with $|X| \geq 3$. Then every quadratic TM-algebra on X is a BCI-algebra.

Proof: It is direct from Definition (2.7) and by Theorem (2.10) [11].

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4. Relation between some algebras.

Theorem 4.1 Every B-algebra is BG-algebra**.**

Proof: It is an instantaneous follower Theorem (2.2) [5].

Because of the following result, the vice versa of theorem (4.1) is not true by Example (2.4) $[(0 * 2) * 1] = (2 * 1) = 2$

and $[0 * (1 * (0 * 2))] = [0 * (1 * 2)] = (0 * 1) = 1$

imply $[(0 * 2) * 1] \neq [0 * (1 * (0 * 2))]$.

Theorem 4.2 Every BG-algebra is BH-algebra.

Proof: By Proposition (2.8) [5].

Because of the following result, the convers of Theorem (4.2) is not true by example (2.6)

$[(2 * 3) * (0 * 3)] = 1 \neq 2.$

Theorem 4.3 Every TM-algebra is BH-algebra.

Proof: $(x * 0) = x$ By definition (TM-algebra)

$(x * x) = 0$ By proposition (2.9)

(For all x and $y \in X$) $((y * x) = 0$ and $(x * y) = 0 \rightarrow x = y)$. By Proposition (2.4) (VI) [11].

Because of the following result, the convers of Theorem (4.3) is not true Example (2.6)

$[(0 * 1) * (0 * 2)] = 3 \neq 2 = (2 * 1)$.

Theorem 4.4 Every TM-algebra is B-algebra

Proof: By Definition (2.7) and Proposition (2.3) [11]

Let $(X, * ,0)$ is TM-algebra

(For all x, y *and* z are elements of X)

$$((x * y) * z = (x * z) * y) \quad (4.1)$$

Put in $y = (e * y)$ in (4.1)

$= (x * (z * (e * y))$

However, it is B-algebra.

For example, suppose that $(X; * ,0)$ is algebra, Where $X = \{0,1,2,3,4,5\}$

*	0	1	2	3	4	5
0	0	2	2	3	4	5
1	1	0	1	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Because of the following result, the convers of Theorem (4.4) is not true

Since $(1 * 2) * (1 * 3) = 2 * 4 = 3 \neq 5 = (3 * 2)$.

Theorem 4.5 Every B-algebra is BH-algebra.

Proof: By definition TM-algebra and Lemma (2.3) [4].

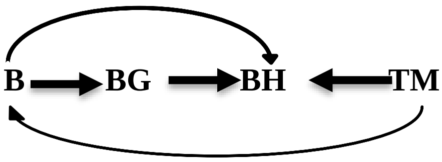
Because of the following result, the vice versa of theorem (4.7) is not true by Example (2.6)

$[(1 * 2) * 3] = (0 * 3) = 2$

and $[1 * (3 * (0 * 2))] = [1 * (3 * 0)] = (1 * 3) = 0$

imply $[(1 * 2) * 3] \neq [1 * (3 * (0 * 2))]$.

We can summarize the diagram as follows:



Conclusion

we present the notion of "quadratic" BH−algebra and TM-algebra, and show that every quadratic BH−algebra and TM-algebra, $(X;*, e)$, $e \in X$ has a product of the form $x * y = x - y + e$, where $x, y \in X$; when X is a field with $|X| \geq 3$ and is BCI-algebra

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