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Applied Lyapunov Stability for Some Nonlinear Stochastic Differential Equations

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ABSTRACT

In this paper, we applied and explain the stability to some linear and non−linear stochastic differential equations by using the Lyapunov direct second method, after finding the stochastic differential equation which obtained by applying the (Ito-integrated formula) and the quadratic Lyapunov function be taken, we use the Lyapunov theorems to find and explain if the trivial (zero) solution are stochastically stabile (p-stable, mean square stable and stochastically asymptotically stable in the large), then we explain the methods by some examples.

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تطبيق استقرارية ليابونوف على بعض المعادلات التفاضلية التصادفية غير الخطية

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الملخص

تم في هذا البحث دراسة الاستقرارية وتطبيقها على بعض المعادلات التفاضلية التصادفية الخطية وغير الخطية باستخدام الطريقة الثانية المباشرة للعالم ليابونوف ،بعد ايجاد المعادلة التفاضلية التصادفية التي تم الحصول عليها بتطبيق صيغة ايتو التكاملية (Ito-integrated formula) وبفرض ان دالة ليابونوف التربيعية معطاة تم استخدام نظريات ليابينوف لايجاد وتوضيح استقرارية الحل الصفري او مايسمى بالحل التافه (مستقر من الرتبة p وكذلك مربع معدل الاستقرارية والاستقرارية المحاذية في الحجم الكبير)،وتم عرض بعض الامثلة لتوضيح الطريقة.

الكلمات المفتاحية: الاستقرارية ، المعادلة التفاضلية التصادفية (العشوائية)، دالة ليابونوف.

Introduction

Studying and applied stochastic differential equations (SDE) is a nature field of research. Different types of SDEs (linear or non- linear) have been used to model different phenomena in various areas, such as non-stable stock prices in finance (Fischer, S., and R.C. Merton [1], the dynamics of some biological systems Jha, S.K., Langmead, C.J [2], filtering such as Kalman filter in navigation control. The stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are close to each other at a specific instant should therefore remain close to each other at all subsequent instants Lawrence C. E [3], the scientist Lyapunov in [4], introduced the new concept of stability in a dynamical system. Since this time, the concept of stability has been studied widely in different senses, Hu, L., Mao, X., & Yi, S. [5], investigated different types of stabilities for stochastic differential equation. Erkan Nane and Yinan Ni [6] are studying and extending the stability for the moments of SDES, Ayman M. Elbaz, William L. Roberts [7] studied the stability of turbulent (linear and non-linear) systems by Lyapunov method approach.

In this paper we use the Ito-integral formula for linear and nonlinear stochastic differential equation after assuming the quadratic Lyapunov function be given in order to applied the stability theorems (Lyapunov second direct method). We explain the methods by introducing some examples.

Suppose $\{x(t)\}$ satisfies the solution of the following stochastic differential equation

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)dW(t), t \geq 0 \qquad (1)$$

Where $N(x(t),t) \in$ R, $N(x(t),t) \in$ R is measurable functions, with $X(0) = x_0$ and $W(t)$ is the standard Brownian process.

The integrating form of eq. (1) which is their solution, is:

$$x(t) = x(0) + \int_0^t N(X(s),s)ds + \int_0^t M(X(s),s)\, dW(s) \qquad (2)$$

suppose that at any initial value $x_t(0) = x_0 \in R^n$, there correspond a unique global solution denoted by $X(t;t_0;x_0)$.

Then equation (1) has the (zero (trivial) solution or equilibrium position) $x_t(0) \equiv 0$ corresponding to the given initial value $x_t(0) = 0$.

Definition (1): [8]

Assume that K denote the family of all continuous non-decreasing functions μ where $\mu: R_+ \to R_+$ such that if r and h are positive numbers, $\mu(0) = 0$ and $\mu(r) > 0$, let $V(x,t)$ be continuous function define on $S_h \times [t_0,\infty]$ $where\ S_h = \{x \in R^n: |x| < h\}$, hence the function $V(x,t)$ is said to be positive-definite if $V(0,t) \equiv 0$ and, for some $\mu \in K$, $V(x,t) \geq \mu(|x|)$ for all $(x,t) \in S_h \times [t_0,\infty]$.

Also, it is said to be negative-definite if (− V)is positive-definite.

Definition (2): [9], [10]

If for every pair of (ε, r) where $\varepsilon \in (0,1)$ and $r > 0$ there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

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$$P\{|x(t;t_0,x_0)| < r \; for \; all \; t \geq t_0\} \geq 1-\varepsilon \quad (3)$$

whenever $|x_0| < \delta_0$, then the trivial solution of equation (1) is stochastically stable or stable in probability. Otherwise, it is said to be unstable stochastically.

Definition (3): [9], [10]

If the trivial solution is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\left\{\lim_{n\to\infty} x(t;t_0,x_0) = 0\right\} \geq 1-\varepsilon$$

whenever $|x_0| < \delta_0$, then the trivial solution of equation (1) is asymptotically stable stochastically.

Also, if it is stochastically stable and for all $x_0 \in R^d$

$$P\left\{\lim_{n\to\infty} x(t;t_0,x_0) = 0\right\} = 1$$

Then the trivial (zero) solution of the equation (1) is asymptotically stable stochastically in the large.

Definition (4): [9]

The trivial solution of

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)dW(t), t \geq 0$$

for some $p > 0$ is called p-stable if for each $\in > 0$ there exists $\delta > 0$ such that $E\|x(t,\emptyset)\|^p < \in, t \geq 0$ provided that $\|\emptyset\|_1^p < \delta$.

Theorem (1): (Lyapunov theorem)_[8], [10]

(i) The trivial(zero) solution is said to be stable, if we find a positive-definite function $V(t,X_t) \in C^{1,1}(S_h \times [t_0,\infty]; R_+)$ such that

For all $(x,t) \in S_h \times [t_0,\infty]$.

$$\dot{V}(x,t) = V_t(t,X(t)) + V_x(t,X(t))f(t,X(t)) \leq 0 \quad (4)$$

(ii) The trivial(zero) solution is called asymptotically stable, if there exists a positive-definite decrescent function $V(t,X_t) \in C^{1,1}(S_h \times [t_0,\infty],, R_+)$ such that the derivative of $V(t,X_t)$ is negative-definite.

Definition 5: [11]

The trivial solution of the following system

$$dx(t) = f(x)dt + h(x)dw(t) \quad (5)$$

is said to be asymptotically mean square stable on the interval $[0,\infty)$ if it is stable and moreover,

$$\lim_{t\to\infty} E^{(1)}\left[\|X(t)\|^2\right] = 0 \quad (6)$$

That is it satisfies the following limitations in the neighborhood of the point $0 \in R^m$:

$$\lim_{t\to\infty} E^{(2)}[x(t)] = \lim_{t\to\infty} E^{(1)}\{X(t)X^T\} = 0 \quad (7)$$

Theorem (2): [8], [9]

i): If we have a positive-definite function $V(y,t) \in C^{2,1}\left(S_h \times [t_{0,}\infty), R_+\right)$ such that, $LV(y,t) \leq 0 \; for$ all $(y,t) \in S_h x(t_0,\infty)$, then the (zero) trivial solution equation (1) is stochastically stable.

ii)If there exists a decrescent function $V(y,t) \in C^{1,2}\left(S_h \times [t_{0,}\infty), R_+\right)$, then the trivial(zero) solution of the given equation is asymptotically stable stochastically if $LV(y,t)$ is negative-definite.

iii)If there exists a decrescent radially unbounded function $V(y,t) \in C^{1,2}\left(R^n \times [t_{0,}\infty), R_+\right)$, then the simple zero solution of the equation (1) is asymptotically stable stochastically in the large if such that $LV(y,t)$ is negative-definite.

II. PREREQUISITES AND RESULTS:

Suppose we have the quadratic Lyapunov function $V(X_t)$ is given

$$V(X_t) = X_t^T Q X_t$$

Where Q is an $m \times m$ symmetric positive definite matrix.

To applied and use the Lyapunov stability for stochastic differential equation:

let $F\big(t,X(t)\big)$ be a smooth function and set $F\big(t,X(t)\big) = V(t,X_t)$ and suppose that it satisfies the existence of solution of equation (1), then we can write it by using Ito - formula as:

$$dV(t,x_t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}N(,t,x_t) + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}M(t,x_t)^2\right)dt + \frac{\partial V}{\partial x}M(t,x_t)dW_t \quad (8)$$

or we can write it as:

$$dV(t,x_t) = LV(t,x_t)dt + \frac{\partial V}{\partial X}M(t,x_t)dW_t \quad (9)$$

The function $LV(X_t) \leq 0$ for stochastic differential equation is equivalence with $\ddot{V}(X_t) \leq 0$ for deterministic equation.

1: Nonlinear case: suppose we have the following equation

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$dV(t,X_t) = LV(t,x_t)dt + \frac{\partial V}{\partial X} M(t,x_t)dW_t$ where $V(X_t) = X_t^T Q X_t$

where $LV(t,x_t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} N(,t,x_t) + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} M(t,x_t)^2\right)$

since,

$$\frac{\partial V}{\partial t} = V_t(t, X(t)) = 0$$

,

$$\frac{\partial V}{\partial X} = V_x(t, X(t)) = 2QX_t^T$$

and

$$\frac{\partial^2 V}{\partial X^2} = V_{xx}(t, X(t)) = 2Q$$

then

$$dV(t,x_t) = [2X_t^T QN(t,X) + M(t,x_t)^T QM(t,x_t)]dt + [2X_t^T Q\,]M(t,x_t)]dW_t$$

That is:

$$LV(x_t) = [(2X_t^T QN(t,X) + M(t,x_t)^T QM(t,x_t)\,] \quad (10)$$

since Q is symmetric matrix and $N(t,X)$ is smooth function, we can write equation (10) as:

$$LV(x_t) = X_t^T QN(t,X) + N(t,x_t)^T QX_t + M(t,x_t)^T QM(t,x_t) \quad (11)$$

which is equivalence with

$$LV(t,x_t) = V_t(t,X(t)) + V_x(t,X(t))N(t,X) + \frac{1}{2} trace M(t,x_t)^T V_{xx}(t,X(t))M(t,x_t)$$

Stochastically asymptotically stable in the large:

From the theorem we need to prove that $LV(x_t)$ is negative-definite in neighborhood of $x_t = 0$ for $t \geq t_0$.

Since $dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t)^T Q(X_t + dX_t) - X_t^T QX_t$

then

$dV(X_t) = [X_t^T + N(t,x_t)^T dt + M(t,x_t)^T dw_t\,]Q[X_t + N(t,x_t) + M(t,x_t)dw_t] - X_t^T QX_t$

$= X_t^T QX_t + X_t^T QN(t,X_t)\,dt + x_t^T QM(t,X_t)\,d w_t + N(t,xt)^T dtQX_t + N(t,Xt)^T dtQN(t,X_t)dt + N(t,Xt)^T dtQN(t,x_t)dt + N(t,x_t)^T dtQM(t,x_t)dw_t + M(t,x_t)^T dw_t Qx_t + M(t,x_t)^T dw_t QM(t,x_t)dw_t - x_t^T Qx_t$

By using the rules $dt.dt = dt.dW_t = dw_t.dt = 0, dw_t.dw_t = dt$

Then We get:

$$dV(x_t) = x_t^T QN(t,x_t)dt + x_t^T QM(t,x_t)dw_t + N(t,x_t)^T dtQx_t + M(t,x_t)^T dw_t Qx_t + M(t,x_t)^T QM(t,x_t)dt$$

By taking the expectation for both sides, and since $\{W_t\}$ is wiener process which have the property $E(W_t) = 0$, then we get

$$E\{dV(x_t)\} = x_t^T QN(t,x_t)dt + N(t,x_t)^T QX_t d_t + M(t,x_t)^T QM(t,x_t)dt = LV(x_t)dt\,.$$

$-LV(x_t) \geq KV(X_t)$; $K = const\,.$

$\frac{d}{dt}E\{V(X_t)\} \leq -KE\{V(X_t)\}$, or $\frac{dE\{V(X_t)\}}{E\{V(X_t)\}} \leq -Kdt$

Then $lnE\{V(X_t)\} \leq -Kt$

$E\{V(X_t)\} \leq \exp(-Kt)$.

and since

$$\lim_{t\to\infty} E^2\{X_t\} = \lim_{t\to\infty} E\{X_t X_t^T\}$$

$$\lim_{t\to\infty} E^2\{X_t\} = \lim_{t\to\infty} \exp(-2Kt) = \lim_{t\to\infty} \exp(-\infty) = 0$$

Therefore equation (12) is asymptotically stable in large, and the trivial solution is unstable if $LV(x_t)$ is positive-definite in some neighborhood of $X_t = 0$.

2: linear stochastic system differential equation:

Suppose we have the following linear system stochastic differential equation

$$dx_t = \propto x_t dt + bx_t dw_t \quad t \geq 0 \quad (12)$$

where $\propto$, b are m×m constant matrices, her $N(x(t),t) = \propto x_t$ and $M(x(t),t) = bx_t$

applying equation (11), we get

$$LV(x_t) = {x_t}^T \propto^T Qx_t + {x_t}^T Q \propto x_t + {x_t}^T b^T Qbx_t$$

In the same method for nonlinear stochastic differential equation, we compute the Lyapunov function:

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$dV(x_t) = V(x_t + dx_t) - V(x_t)$
$= (x_t + dx_t)^T Q(x_t + dx_t) - {x_t}^T Q x_t$
$= [{x_t}^T + (dx_t)^T] Q(x_t + dx_t) - {x_t}^T Q x_t$
$= [{x_t}^T + (\propto x_t)^T dt + (\beta x_t)^T d\beta_t] Q[x_t + \{(\propto x_t)dt + \beta x_t d\beta_t\}] - [{x_t}^T Q x_t]$
$= {x_t}^T Q x_t + {x_t}^T Q(\propto x_t)dt + {x_t}^T \beta x_t d\beta_t + (\propto x_t)^T dt Q x_t + (\propto x_t)^T dt Q(\propto x_t)dt + (\propto x_t)^T dt Q \beta x_t d\beta_t + (\beta x_t)^T d\beta_t Q x_t + (\beta x_t)^T d\beta_t Q(\propto x_t)dt - {x_t}^T Q x_t$

By applying the rules $dt.dt = dt.d\beta_t = d\beta_t.dt = 0, d\beta_t.d\beta_t = dt$ .And taking expectation With (E $(\beta_t) = 0$)

$$E\{dV(x_t)\} = x_t^T Q(\propto x_t)dt + (\propto x_t)^T Q x_t dt + (\beta x_t)^T Q \beta x_t dt = LV(x_t)dt$$

Then $LV(x_t) \le 0$ if and only if

$$[{x_t}^T \propto^T Q x_t + {x_t}^T Q \propto x_t + {x_t}^T b^T Q b x_t] \le 0 \qquad (13)$$

After we find the values that satisfies the above equation (13), this explains how to find the stability of the given equation.

For asymptotically stability we must have

$$\lim_{t\to\infty} E^2\{X_t\} = 0$$

Examples: we give some examples in order to apply and explain the methods.

Example (1): let $\{X_t\}$ satisfies the solution of the following non-linear stochastic differential equation

$$dX_t = (a{X_t}^n + bX_t)dt + cX_t d\mathcal{W}_t \qquad (14)$$

Where a, b, c are constants, $\mathcal{W}_t$ *is the wiener process* .

Determine the Lyapunov function and the stability.

Solution: Here $N(t, X_t) = (a{X_t}^n + bX_t)$; $M(t,X_t) = cX_t$

Then from equation (8), we have

$LV(X_t) = X_t^T Q N(t, X_t) + N(t, Xt)^T Q X_t + M(t, X_t)^T Q M(t,X_t)$

Or

$LV(X_t) = X_t^T Q(a{X_t}^n + bX_t) + (a{X_t}^n + bX_t)^T Q X_t + (cX_t)^T Q (cX_t)$

Since Q=1, then

$$LV(t, x_t) = 2ax^{n+1} + 2bx_t^2 + c^2 x_t^2$$

To find the Lyapunov function, let $V(t, X_t) = V(X_t) = X_t^T Q X_t$, then

$dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t)^T Q(X_t + dX_t) - X_t^T Q X_t$

$dV(X_t) = x_t^T Q(ax_t^n + bx_t)dt + x_t^T Q c x_t dw_t + (ax_t^n + bx_t)^T dt Q x_t + (cx_t)^T dw_t Q x_t + (cx_t)^T Q c x_t dt = (ax^{n+1} + bx_t^2)dt + cx_t^2 dw_t + (ax_t^{n+1} + bx_t^2)dt + cx_t^2 dw + c^2 x_t^2 dt] = (ax^{n+1} + bx_t^2 + ax_t^{n+1} + bx_t^2 + c^2 x^2)dt + 2cx_t^2 dw = (2ax^{n+1} + 2bx_t^2 + c^2 x_t^2)dt + 2cx_t^2 dw$

Then $dE(V(X_t)) = (2a{x_t}^{n+1} + 2bx_t^2 + c^2 x_t^2)dt = LV(t, x_t)dt$

To apply Theorem (2), we need to show that there exists a neighborhood of the zero point for the equation:

$2a{x_t}^{n+1} + 2bx_t^2 + c^2 x_t^2 \le 0.$

This holds if and only if the following inequality is satisfied $x_t \le (\frac{-(2b+c^2)}{2a})^{\frac{1}{n-1}}$, n≠1. Thus, to obtain $LV(t, X_t) < 0$, x_t must satisfies the inequality $\le (\frac{-(2b+c^2)}{2a})^{\frac{1}{n-1}}$; n≠1, at each point also if $x_t = 0$ *we, get* $LV(0) = 0$. Therefore, we conclude that there exists a neighborhood in which the function $LV(X_t) = 2ax^{n+1} + 2bx_t^2 + c^2 x_t^2$ is negative definite. So, the trivial(zero) solution $x_t = 0$ of considered equation is asymptotically mean square stable on the interval $[0, \infty)$ since $\lim_{k\to\infty} E^2\{X_t\}$ *equal to zero*, i.e.

Since $-LV(x_t) \ge KV(X_t)$; *where K is const* .

$\frac{d}{dt}E\{V(X_t)\} \le -KE\{V(X_t)\}$, or $\frac{dE\{V(X_t)\}}{E\{V(X_t)\}} \le -Kdt$

Then by integration, $lnE\{V(X_t)\} \le -Kt$ therefore $E\{V(X_t)\} \le \exp(-Kt)$ .and since

$\lim_{t\to\infty} E\{X_t^2\} = \lim_{t\to\infty} E\{X_t X_t^T\}$, we get $\lim_{t\to\infty} E\{X_t^2\} = \lim_{t\to\infty}(-2Kt) = 0.$

Example (2): suppose we have the following stochastic differential equation:

$$dX_t = 3X_t dt + \exp(t)^2 dw_t \qquad (15)$$

Then $LV(X_t) == (6X_t^2 + exp2t^2)$

To find the Lyapunov function, let $V(X_t) = X_t^T Q X_t$, since

$dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t)^T Q(X_t + dX_t) - X_t^T Q X_t$

or,

$$dV(X_t) = X_t^T(3X_t)dt + X_t^T \exp t^2 dw_t + (3X_t)^T X_t dt + \exp(t^2) X_t dw + (\exp t^2)^T \exp(t^2) dt$$

By taking the expectation, we get

$E(dV(X_t)) = 3X_t^2 dt + 3X_t^2 dt + exp2t^2 dt = (6X_t^2 + exp2t^2)dt$

That is $E(dV(X_t)) = LV(X_t)dt$

Then the stability condition is $(6X_t^2 + exp2t^2) \le 0$ which is hold if and only if

$$X_t \le \left(\frac{-(e^{2t^2})}{6}\right)^{1/2}$$

For asymptotically stochastically stable we need to show that if the following condition satisfied

$$\lim_{t\to\infty} E^2\{X_t\} = 0$$

since

$\frac{(dEV(X_t))}{E(V(X_t))} = (6X_t^2 + exp2t^2)dt$

$\text{Ln } E(V(X_t)) = \int_0^t (6X_s^2 + exp2s^2)ds$

$E(V(X_t)) = \exp\left(\int_0^t (6X_s^2 + exp2s^2)ds\right)$, then

$\lim_{t\to\infty} E^2\{X_t\} = \lim_{t\to\infty} \exp\left(2(\int_0^t (6X_s^2 + exp2s^2)ds)\right) \ne 0$, then the stochastic differential equation is not asymptotically stochastically stable.

Ex: (3): (linear model), Let we have the following linear stochastic differential equation:

$dX_t = 2X_t dt + 3X_t\, dW_t$

(16)

Then $LV(x_t) = 13x_t^2$. (Where Q=1), the quadratic function $V(X_t) = X_t^T Q X_t$, with Q=1

Then,

$$dV(x_t) = V(x_t + dx_t) - V(x_t) = (x_t + dx_t)^T Q(x_t + dx_t) - x_t^T Q x_t$$

$$dV(x_t) = x_t^T Q(2x_t)dt + x_t^T Q(3x)dw_t + (2x_t)^T dt Q x_t + 3x dw_t Q x_t + (3x)^T Q(3x)dt$$

$= 4x_t^2 dt + 9x^2 dt + 6x^2 dw$

$= 13x_t^2 dt + 6x^2 dw$

$\therefore LV(x_t) = 13x_t^2 \ge 0$ for all values of x_t, then the trivial(zero) solution of equation (16) is non-stable and also not asymptotically stable.

Ex: (4): suppose we have the following non-linear (Square root S.D.E)

$dx_t = \propto (\theta - X(t))dt + \gamma\sqrt{X(t)}dw(t)$

(17)

Hence $LV(X_t) = [(2\alpha\theta + \gamma^2)X(t) - 2X(t)^2]$.

To find Lyapunov function, let $V(X_t) = X_t^T Q X_t$, then

$dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t)^T Q(X_t + dX_t) - X_t^T Q X_t$

Then

$$dV(x_t) = X_t^T Q(\alpha(\theta - X(t))dt + X_t^T Q\gamma\sqrt{X(t)}dw + (\alpha(\theta - X(t))^T dt Q x_t + (\gamma\sqrt{X(t)})^T dw_t Q X_t + (\gamma\sqrt{X(t)})^T Q\gamma\sqrt{X(t)}dt$$

$$E(dV(X_t)) = X_t^T(\alpha(\theta - X(t))dt + (\alpha(\theta - X(t))^T X_t dt + (\gamma\sqrt{X(t)})^T \gamma\sqrt{X(t)}\, dt$$

$= 2X_t(\alpha(\theta - X(t))dt + \gamma^2 X(t)dt$

$= [(2\alpha\theta + \gamma^2)X(t) - 2X(t)^2]dt$

Then the trivial solution of equation (20) is stable for all $X(t) > \frac{2\alpha\theta - \gamma^2}{2}$ and unstable if $X(t) < \frac{2\alpha\theta - \gamma^2}{2}$.

III. CONCLUSION AND FUTURE WORKS:

We know that the trivial solution is said to be stable if the derivative of Lyapunov function is less than or equal to zero, while if it is only negative-definite then it is asymptotically stable. To find the stability of stochastic differential equation we use the function $LV(X_t) \le 0$ which is equivalence with the inequality $\dot{V}(X_t) \le 0$ for deterministic equation ,we explain the stability condition for some nonlinear stochastic differential equation by using the direct method (lyapunov direct method), also we explain asymptotically stable in the large not almost this condition is satisfies, that is if the trivial solution is asymptotically stable but not asymptotically stable in large by the fact if the limit is not equal to zero. we explain the methods by several examples.

As a future studies one can study the stability (direct method) for some nonlinear (harmonic or exponential) stochastic differential equation by using stratonovich formula for their solution compare it with Ito formula

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