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Bifurcation Analysis in a Discrete-Time Prey-Predator System with Crowly- Martin Functional Response

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ABSTRACT

In this paper, a discrete time prey-predator system with Crowly- Martin functional response was studied. The fixed points of the model are obtained, and their stability is analyzed. Further existence of bifurcation analysis at each fixed point and Hopf bifurcation are demonstrated. Numerical simulation show that the model perhaps has more complex dynamical behaviors, such as the period-5,10,20 and 35 orbits, including the periodic doubling bifurcation in period-2,4,8 and 16 orbits and chaotic set.

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تحليل التشعب في نظام زمني منفصل للفريسة–المفترس مع دالة استجابة Crowly-Martin

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الملخص

في هذا البحث، تمت دراسة نظام زمني منفصل للفريسة-المفترس مع دالة استجابة (Crowly-Martin). حيث تم الحصول على النقاط الثابتة للنموذج وتم تحليل استقراريتها. مزيد من تحليل وجود التشعب في كل نقطة ثابتة وتشعب هوبف تم توضيحها. تظهر المحاكاة العددية أن النموذج ربما يحتوي على سلوكيات ديناميكية أكثر تعقيدًا، مثل مدارات الفترة ٥, ١٠, ٢٠ و٣٥ بما في ذلك التشعب الدوري المضاعف في الفترة ٢, ٤, ٨ و ١٦ والمجموعة الفوضوية (chaotic set).

1. Introduction

In theory of population dynamics, includes two types of mathematical models, the discrete time models and countious time model. The simplest countinous time model, first introduced by Verhulst [1] and later further attention by Pearl and Reed [2]. The result of discrete time models are richer compared than continous time models, becouse in the discrete time models is obtained accurate numerical simulations result. Morover, discrete time models are more appropriate than the continuous time models. Authors have already focused on the Prey-predator models. For example, the stability and the existence of periodic solution of the prey-predator models studied in [3], [4], [5], [6] and [7].

The functional response is the rate which an animal consumes prey per unit time. Holling, organized the functional response into three types: Holling type I, II and III [8] and [9]. The intrinsic charateristic of these function, predators benefit when there are more prey in inviroment, which is true in many predator prey interaction. The type of functional response are summarized in Table 1

The dynamic of discrete-time model is still an important topic of reserch becouse appears in a variety of application such as ecology and mathematical biology. In ecology, plant-herbivor or prey-predator models can be written as discrete time model. A plant-herbivor model refer to investigate interaction between a plant types and herbivor types. Discrete models described by difference equations for interacting populations are of significant attention to biologists. Discrete-time models are more appropriate than continuous time models when the population have non-overlap generation. This definitely occurs for population that have one-year life like insect. A discrete-time model can display more complicated dynamical behaviors than a continuous-time model of the same type.

Lotka-Volterra is one of the most known models to the investigate dynamic behavior of the predator-prey system. A pair of first-order nonlinear differential equation, often used to explain the dynamics of biological system, the interaction between them are of two types, one a predator and the other prey.

Table 1: Holling types of functional respones

Holling type	Definition	Generalized form	Applications
I	$\Phi(x) = \mu x$		
II	$\Phi(x) = \dfrac{\mu x}{a + x}$		[10]
III	$\Phi(x) = \dfrac{\mu x^2}{a + x^2}$	$\Phi(x) = \dfrac{\mu x^2}{1 + bx + ax^2}, (b > -2\sqrt{a})$	[11] and [12]
IV	$\Phi(x) = \dfrac{\mu x}{a + x^2}$	$\Phi(x) = \dfrac{\mu x}{1 + bx + ax^2}, (b > -2\sqrt{a})$	[13] and [14]
Beddington DeAgelis	$\Phi(x, y) = \dfrac{\mu x}{1 + by + ax}$		[15]

However, authors don't much describe about the dynamical behavior of prey-predator models, which include bifurcation and chaos phenomena for the discrete time models. Liu and Xiao [16] and He and Lai [17]. Obtained the flip bifurcation by using center manifold theorem and bifurcation theory.

initially, Lotka (1925) [18] and Volterra (1926) [19] have described prey-predator system, always be in the following form

$$\begin{cases} \dot{x} = xq(x) - yp(x,y) \\ \dot{y} = y(\delta p(x,y) - \gamma) \end{cases} \quad (1)$$

where $x(t)$ show the prey density and $y(t)$ show the predator density, the function $p(x,y)$ describes the predator functional response and δ is the conversion rate of prey into predators and γ predator death rate. In this paper is used Crowly-Martin of functional response $p(x,y) = \frac{mx}{(1+c_1x)(1+c_2y)}$, $q(x) = 1 - x$, then Eq(1) becomes the following

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{mxy}{(1+c_1x)(1+c_2y)} \\ \dot{y} = y(\dfrac{\delta mxy}{(1+c_1x)(1+c_2y)} - \gamma) \end{cases} \quad (2)$$

where c_1, c_2 and m are nonnegative paramers. The dynamical behavior of model (2) in the mathematics literatures for rare cases have happened. If $c_2 = 0$ of the model (2) have been investigated by many writer [20],[21] and [22] and it was demonstrated that only stable equilibrium point or limit cycle are included in dynamic. Applying Euler scheme to model (2) we obtain the descrite time system

$$\begin{cases} x_{n+1} = x_n + c_5x_n\left(1 - x_n - \dfrac{y_n}{(1+c_1x_n)(1+c_2y_n)}\right) \\ y_{n+1} = y_n + c_5y_n\left(\dfrac{c_3x_n}{(1+c_1x_n)(1+c_2y_n)} - c_4\right) \end{cases} \quad (3)$$

where c_5 is the step size. We will focus on the discrete time model (3) in the closed first quadrant $\mathbb{R}_+^2$ on the (x,y) plane. Our goal in this paper is demonstrated complex dynamical behavior. Furthere, it can be shown that the model (3) undegoes flip bifurcation and Hopf bifurcation by using center manifold theorem and bifurcation theory.

The arranging of this paper is as follows: in Section 2, we study the existence three fixed point of the model (3) and stability. Section 3, mainly analyze the flip bifurcation and Hopf bifurcation of the model (3) in the interior $\mathbb{R}_+^2$. Section 4, we present the numerical simulation, which not only show our results but also explain the complex dynamical behavior, includig bifurcation diagrams and different types of attractors. A brief conclusion is given section 5.

2. Fixed points and their stability analysis

Consider the dynamical behavior of the model (3) in the closed first quadrant $\mathbb{R}_+^2$ on the (x,y) plane. We will try to find it is fixed point and study their stability. To determine the fixed point of model (3) satiesfy the following equation

$$\begin{cases} x = x + c_5x\left(1 - x - \frac{y}{(1+c_1x)(1+c_2y)}\right) \\ y = y + c_5y\left(\frac{c_3x}{(1+c_1x)(1+c_2y)} - c_4\right) \end{cases} \quad (4)$$

By solving the equation (4), we have the following Lemma.

Lemma 2.1

(i) For all positive parameter, the model (4) always has two fixed point $E_0(0,0)$ and $E_1(1,0)$

(ii) The positive fixed point $E_2(x^*,y^*)$ exists in the Int. $\mathbb{R}_+^2$ if there is a positive solution to the following set of equation.

$$\begin{cases} 1 - x - \frac{y}{(1+c_1x)(1+c_2y)} = 0 \dots\dots (a), \\ -c_4 + \frac{c_3x}{(1+c_1x)(1+c_2y)} = 0 \dots\dots (b). \end{cases} \quad (5)$$

From Eq. 5 (b) we can get, $y^* = \frac{(c_3 - c_1c_4)x^* - c_4}{c_2c_4(1+c_1x^*)}$ and $y^* > 0$ such that $\frac{c_4}{c_3 - c_1c_4} < x^* < 1$, by substituting the value of y^* and solving for x^* we get the polynomial equation $k(x)$ where

$$k(x) = c_1c_2c_3x^3 + c_2c_3(1-c_1)x^2 + (c_3 - c_1c_4 - c_2c_3)x - c_4 = 0 \quad (6)$$

We have $k(0) = -c_4 < 0$ and $k(1) = c_3 - c_1c_4 - c_4 > 0$, by intermediate value theorem there exist at least positive $x^* \in (0,1)$ such that $k(x^*) = 0$.

Next, we will study stability of fixed point E_0, E_1 and E_2 of model (3), by introducing the following Lemma.

Lemma 2.2 [16] Let $F(\lambda) = \lambda^2 - B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ $(or\ |\lambda_1| > 1\ and\ |\lambda_2| < 1)$ if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;

(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0,2$;

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(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2|$ if and only if $B^2 - 4C < 0$ and $C = 1$;

Let λ_1 and λ_2 be the two roots of jacobian matrix, which are called eigenvalues of the fixed point (x, y). We recall some definitions of topological types for a fixed point (x, y). A fixed point (x, y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, (x, y) is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$), and (x, y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$,.

Jacobian matrix for the fixed point $E_0(0,0)$ is given by

$$J(E_0) = \begin{bmatrix} 1 + c_5 & 0 \\ 0 & 1 - c_4c_5 \end{bmatrix}$$

The eigenvalue of the jacobian matrix are $\lambda_1 = 1 + c_5$ and $\lambda_2 = 1 - c_4c_5$, we have the following result.

Lemma 2.3 For the fixed point E_0, we have the following conditions

(i) E_o is a saddle if $0 < c_5 < \frac{2}{c_4}$;

(ii) E_0 is a source if $c_5 > \frac{2}{c_4}$;

(iii) When $c_5 = \frac{2}{c_4}$ so one of the eigenvalues $\lambda_2 = -1$, we have a non-hyperbolic fixed point at $E_0(0,0)$ which is repelling

Proof. In the present case $c_4c_5 = 2$, we calculated lower the center manifold at E_0 and the dynamics reduced to the center manifold to determine the orbit around E_0 in the first quadrant $\mathbb{R}^2_+$.

$$\begin{aligned} f_1(x,y) = \quad & x + c_5\left(1 - x - \frac{y}{(1 + c_1x)(1 + c_2y)}\right) = (1 + c_5)x - c_5x^2 - c_5xy + c_1c_5x^2y + c_2c_5xy^2 \\ & -c_1^2c_5x^3y - c_1c_2c_5x^2y^2 - c_2^2c_5xy^3 + O(5), \\ g_1(x,y) = \quad & y + c_5y\left(\frac{c_3x}{(1 + c_1x)(1 + c_2y} - c_4\right) = (1 - c_4c_5)y + c_3c_5xy - c_1c_3c_5x^2y - c_2c_3c_5xy^2 \\ & +c_1^2c_3c_5x^3y + c_1c_2c_3c_5x^2y^2 + c_2^2c_3c_5xy^3 + O(5). \end{aligned} \quad (7)$$

Then the equation (7) is Jordan normal form, we have the followin model

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + c_5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -c_5x^2 - c_5xy + c_1c_5x^2y + c_2c_5xy^2 + O(3) \\ c_3c_5xy - c_1c_3c_5x^2y - c_2c_3c_5xy^2 + O(3) \end{pmatrix}$$

Thus, the center manifold is defined by

$$y = h^c(x) = \alpha x^2 + \beta x^3 + O(4) \equiv 0$$

Then the dynamic on the center manifold is defined by the following map

$$x \mapsto (1 + c_5)x - c_5x^2 + O(3)$$

which shows that $x = 0$ is repelling. Therefore, first derivate of right side at $x = 0$ is greater than one [23].

Jacobian matrix for the fixed point $E_1(1,0)$ is given by

$$J(E_1) = \begin{bmatrix} 1 - c_5 & -\dfrac{c_5}{1 + c_1} \\ 0 & 1 + c_5\left(\dfrac{c_3 - c_4(1 + c_1)}{1 + c_1}\right) \end{bmatrix}$$

The eigenvalue of the jacobian matrix are $\lambda_1 = 1 - c_5$ and $\lambda_2 = 1 + c_5\left(\frac{c_3 - c_4(1+c_1)}{1+c_1}\right)$, we have the following result.

Lemma 2.4 For the fixed point E_1, the are at least four different topological classification hold:

(i) E_1 is a sink if $c_3 < c_4(1 + c_1)$ and $0 < c_5 < min\{2, \frac{2(1+c_1)}{c_4(1+c_1)-c_3}\}$;

(ii) E_1 is a source if $c_5 > 2$ and $c_3 > c_4(1 + c_1)$ or ($c_3 < c_4(1 + c_1)$ and $c_5 > max\{2, \frac{2(1+c_1)}{c_4(1+c_1)-c_3}\}$)

(iii) E_1 is a saddle if $c_5 < 2$ and $c_3 > c_4(1 + c_1)$ or $(((0 < \frac{2(1+c_1)}{c_4(1+c_1)-c_3} < c_5 < 2)$ or $(2 < c_5 < \frac{2(1+c_1)}{c_4(1+c_1)-c_3}))$ and $c_3 < c_4(1 + c_1))$;

(iv) Whene $c_3 = c_4(1 + c_1)$, so one of the eigenvalue $\lambda_2 = 1$, we have a non-hyperbolic fixed point at $E_1(1,0)$ which is repelling

Proof. In the present case $c_3 = c_4(1 + c_1)$, we calculated lower the center manifold at $E_1(1,0)$ and the dynamics reduced to center manifold to determine the orbit around $E_1(1,0)$ in the first quadrant $\mathbb{R}_+^2$. First, we transform the fixed point $E_1(1,0)$ into the origin by using a linear transformation, i.e. $(x,y) \to (x+1,y)$. Then it become the following map.

$$
\begin{aligned}
f_2(x,y) = \; & x + c_5\left(1 - x - \frac{y}{(1+c_1x)(1+c_2y)}\right) - 1 = (1-c_5)x - \frac{c_5}{1+c_1}y - c_5x^2 - \frac{c_5}{(1+c_1)^2}xy \\
& + \frac{c_2c_5}{1+c_1}y^2 + \frac{c_1c_5}{(1+c_1)^2}x^2y + \frac{c_2c_5}{(1+c_1)^2}xy^2 - \frac{c_2^2c_5}{1+c_1} + O(4), \\
g_2(x,y) = \; & y + c_5y\left(\frac{c_3x}{(1+c_1x)(1+c_2y} - c_4\right) = y + \frac{c_3c_5}{(1+c_1)^2}xy - \frac{c_2c_3c_5}{1+c_1}y^2 - \frac{c_1c_3c_5}{(1+c_1)^3}x^2y \\
& - \frac{c_2c_3c_5}{(1+c_1)^2}xy^2 + \frac{c_2^2c_3c_5}{1+c_1}y^3 + O(4).
\end{aligned}
\tag{8}
$$

Next, by using linear transformation of Equation (8) in Jordan normal form

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{1+c_1} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \frac{1}{1+c_1}v \\ -v \end{pmatrix}
$$

This bring in to the followin

$$
\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1-c_5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\frac{c_5((c_1+c_4)+c_2(1+c_1)^2(c_4-1))}{(1+c_1)^3}v^2 - \frac{c_5(c_4+2c_1+1)}{(1+c_1)^2}uv \\ -c_5u^2 + O(3) \\ \frac{c_4c_5}{(1+c_1)^2}uv + \frac{c_4c_5(c_2(1+c_1)^2+1)}{(1+c_1)^2}v^2 + O(3) \end{pmatrix}
$$

Thus, the center manifold defined by

$$
u = h^c(v) = \alpha v^2 + \beta v^3 + O(4) = -\frac{((c_1+c_4)+c_2(1+c_1)^2(c_4-1))}{(1+c_1)^3}v^2 + O(3).
$$

Then the dynamic on the center manifold id defined by the following map

$$
v \mapsto v + \frac{c_4c_5(c_2(1+c_1)^2+1)}{(1+c_1)^2}v^2 + O(3),
$$

Which see that $v=0$ is repelling. Therefore, first derivative of right side at $x=0$ equal to one and secound derivative is gretar than zero [23]

From the Lemma (2.4) we obtain, for a fixed point $E_1(1,0)$ if $(c_1,c_2,c_3,c_4,c_5) \in \mathfrak{R}$ where

$$
\mathfrak{R} = \{(c_1,c_2,c_3,c_4,c_5) \in (0,\infty) : c_5 = 2, c_3 \neq c_4(1+c_1), c_5 \neq \frac{2(1+c_1)}{c_4(1+c_1)-c_3}, c_i > 0, \mathrm{i} = 1 \dots 5\}
$$

Therefore, a flip bifurcation can appear if parameters change in a small neighborhood of $\mathfrak{R}$. When the $(c_1,c_2,c_3,c_4,c_5) \in \mathfrak{R}$, a model (3) restricted to this center manifold.

Proposition 2.5 Whene $c_5 = 2$, so one of the eigenvalue $\lambda_1 = -1$, we have a non-hyperbolic fixed point at $E_1(1,0)$ which is attracting.

Proof. Same as to the Proof (iv) in Lemma (2.4), for the case, $c_5 = 2$, we can obtain $W^c(1,0): v = h^c(u) \equiv 0$ and

$$
u \mapsto -u - 2u^2 + O(3), \quad as\ u \to 0
$$

which see that $u=0$ is attracting. Therefore, first derivateve of right side at $x=0$ equal to negative one by use this $-2f'''(x) - 3(f''(x))^2$ at $x=0$ is less than zero [23]

Jacobian matrix for the fixed point $E_2(x^*,y^*)$ is given by

$$
J(E_2) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}
$$

Where

$$
j_{11} = 1 + c_5\left(-x^* + (1 - x^*)\frac{c_1x^*}{1+c_1x^*}\right), \qquad j_{12} = -\frac{c_4c_5}{c_3}(1 - c_2(1-x^*)(1+c_1x^*))
$$

$j_{21} = c_3c_5(1 - x^*)\frac{1}{1+c_1x^*}$, $j_{22} = 1 - c_2c_4c_5(1-x^*)(1+c_1x^*)$

Charateristic equation of the jacobian matrix at $E_2(x^*,y^*)$ can be written as

$$F(\lambda) = \lambda^2 - Tr(J(x^*,y^*))\lambda + Det(J(x^*,y^*)) = 0$$

Where

$Tr(J(x^*,y^*)) = 2 - c_5N^*$ and $Det(J(x^*,y^*)) = 1 - c_5N^* + M^*$,

and

$$M^* = c_4(1-x^*)\left(c_2(1+c_1x^*)(2x^*-1) + \frac{1}{1+c_1x^*}\right),$$
$$N^* = x^* + (1-x^*)\left(c_2c_4(1+c_1x^*) - \frac{c_1x^*}{1+c_1x^*}\right)$$

From Lemma 2.2, we obtain the following result.

Lemma 2.6 If $N^* \geqslant 2\sqrt{M^*}$ then $E_2(x^*,y^*)$ of model (3) is

(i) $E_2(x^*,y^*)$ is a sink if $0 < c_5M^* < N^* - \sqrt{N^{*2} - 4M^*}$,

(ii) $E_2(x^*,y^*)$ is a source if $c_5M^* > N^* + \sqrt{N^{*2} - 4M^*}$,

(iii) $E_2(x^*,y^*)$ is a saddle if $N^* - \sqrt{N^{*2} - 4M^*} < c_5M^* < N^* + \sqrt{N^{*2} - 4M^*}$,

(iv) $E_2(x^*,y^*)$ is a non-hyperbolic if $c_5M^* = N^* \pm \sqrt{N^{*2} - 4M^*}$,

Lemma 2.7 If $N^* < 2\sqrt{M^*}$ then $E_2(x^*,y^*)$ of model (3) is

(i) $E_2(x^*,y^*)$ is a sink if $0 < c_5M^* < N^*$,

(ii) $E_2(x^*,y^*)$ is a source if $c_5M^* > N^*$,

(iii) $E_2(x^*,y^*)$ is a non-hyperbolic if $c_5M^* = N^*$

3. Bifurcation Analysis

In this section, we will study the flip bifurcation of the model (3) at positive fixed point $E_2(x^*,y^*)$ by using center manifold theorem and bifurcation theory in [14] and [19]. We set c_5 as a bifurcation parameter.

First, consider model (3) at the positive fixed point $E_2(x^*,y^*)$ with parameter $(c_1,c_2,c_3,c_4,c_5) \in A_1$. The similar arguments can consider the case of A_2.

Taking parameter $(c_1,c_2,c_3,c_4,c_5) = (c_{10},c_{20},c_{30},c_{40},c_{50}) \in A_1$. Further, choosing c_5^* as a bifurcation parameter. We consider a perturbation of model (3) in the following.

From Lemma (2.2), we easily see that one of the eigenvalue of $J(E_2)$ is -1 and the other is neither 1 nor -1, if condition (iv) from Lemma (2.6) hold. Whene the parameter of the model (3) are located in the followin set:

$$A_1 = \{(c_1,c_2,c_3,c_4,c_5) \in (0,+\infty): c_5M^* = N^* + \sqrt{N^{*2} - 4M^*}, \quad N^* > 2\sqrt{M^*}\}$$

or

$$A_2 = \{(c_1,c_2,c_3,c_4,c_5) \in (0,+\infty): c_5M^* = N^* - \sqrt{N^{*2} - 4M^*}, \quad N^* > 2\sqrt{M^*}\}.$$

$$\begin{cases} x_{n+1} = x_n + (c_{50}+c_5^*)x_n\left(1 - x_n - \frac{y_n}{(1+c_{10}x_n)(1+c_{20}y_n)}\right) \\ y_{n+1} = y_n + (c_{50}+c_5^*)y_n\left(\frac{c_{30}x_n}{(1+c_{10}x_n)(1+c_{20}y_n)} - c_{40}\right) \end{cases} \quad (9)$$

where $|c_5^*| \ll 1$. Let $u_n = x_n - y^*$ and $v_n = y_n - y^*$, then fixed point $E_2(x^*,y^*)$ transformed in to the origin. we have

$$\begin{cases} u_{n+1} = u_n + x^* + (c_{50}+c_5^*)(u_n+x^*)\left(1 - u_n - x^* - \frac{v_n + y^*}{(r_1+c_{10}u_n)(r_2+c_{20}v_n)}\right) \\ y_{n+1} = v_n + y^* + (c_{50}+c_5^*)(v_n+y^*)\left(\frac{c_{30}(u_n+x^*)}{(r_1+c_{10}u_n)(r_2+c_{20}v_n)} - c_{40}\right) \end{cases} \quad (10)$$

where $r_1 = 1 + c_{10}x^*$ and $r_2 = 1 + c_{20}y^*$. Expanding model (10) as a tyloar series at $(u,v,c_5^*) = (0,0,0)$ to second order, become

$$\begin{cases} u_{n+1} = a_1u_n + a_2v_n + a_3u_n^2 + a_4u_nv_n + a_5v_n^2 + a_6c_5^* + a_7u_nc_5^* + a_8v_nc_5^* + a_9u_n^2c_5^* \\ \qquad +a_{10}u_nv_nc_5^* + a_{11}v_n^2c_5^* + O(|u_n| + |v_n| + |c_5^*|)^3, \\ v_{n+1} = b_1u_n + b_2v_n + b_3u_n^2 + b_4u_nv_n + b_5v_n^2 + b_6c_5^* + b_7u_nc_5^* + b_8v_nc_5^* + b_9u_n^2c_5^* \\ \qquad +b_{10}u_nv_nc_5^* + b_{11}v_n^2c_5^* + O(|u_n| + |v_n| + |c_5^*|)^3, \end{cases} \tag{11}$$

where

$$\begin{aligned}
a_1 &= 1 + c_{50} - 2c_{50}x^* - c_{50}y^* + 2c_{10}c_{50}x^*y^* + c_{20}c_{50}y^{*2} - 3c_{10}^2c_{50}x^{*2}y^* - 2c_{10}c_{20}c_{50}x^{*2}y^{*2} \\ &\quad - c_{20}c_{50}y^{*3}, \\
a_2 &= -c_{50}x^* + c_{10}c_{50}x^{*2} + 2c_{20}c_{50}x^*y^* - c_{10}c_{50}x^{*3} - 2c_{10}c_{20}c_{50}x^{*2}y^* - 3c_{20}^2c_{50}x^*y^{*2}, \\
a_3 &= -2c_{50} + 2c_{10}c_{50}y^* - 6c_{10}^2c_{50}x^*y^* - 2c_{10}c_{20}c_{50}y^{*2}, \\
a_4 &= -c_{50} + 2c_{10}c_{50}x^* + 2c_{20}c_{50}y^* - 3c_{10}^2c_{50}x^* - 4c_{10}c_{20}c_{50}x^*y^* - 3c_{20}^2c_{50}y^{*2}, \\
a_5 &= 2c_{20}c_{50}x^* - 2c_{10}c_{20}c_{50}x^{*2} - 6c_{20}^2c_{50}x^*y^*, \\
a_6 &= x^* - x^{*2} - x^*y^* + c_{10}x^{*2}y^* + c_{20}x^*y^{*2} - c_{10}^2x^{*3}y^* - c_{10}c_{20}x^{*2}y^{*2} - c_{20}^2x^*y^{*3}, \\
a_7 &= 1 - 2x^* - y^* + 2c_{10}x^*y^* + c_{20}y^{*2} - 3c_{10}^2x^{*2}y^* - 2c_{10}c_{20}x^{*2}y^{*2} - c_{20}^2y^{*3}, \\
a_8 &= -x^* + c_{10}x^{*2} + 2c_{20}x^*y^* - c_{10}x^{*3} - 2c_{10}c_{20}x^{*2}y^* - 3c_{20}^2x^*y^{*2}, \\
a_9 &= -2 + 2c_{10}y^* - 6c_{10}^2x^*y^* - 2c_{10}c_{20}y^{*2}, \\
a_{10} &= -1 + 2c_{10}x^* + 2c_{20}y^* - 3c_{10}^2x^* - 4c_{10}c_{20}x^*y^* - 3c_{20}^2y^{*2}, \\
a_{01} &= 2c_{20}x^* - 2c_{10}c_{20}x^{*2} - 6c_{20}^2x^*y^*, \\
b_1 &= c_{30}c_{50}y^* - 2c_{10}c_{30}c_{50}x^*y^* - c_{20}c_{30}c_{50}y^{*2} + 3c_{10}^2c_{30}c_{50}x^{*2}y^* + 2c_{10}c_{20}c_{30}c_{50}x^*y^{*2} \\ &\quad + c_{20}^2c_{30}c_{50}y^{*3}, \\
b_2 &= 1 + c_{30}c_{50}x^* - c_{10}c_{30}c_{50}x^{*2} - 2c_{20}c_{30}c_{50}x^*y^* + c_{10}^2c_{30}c_{50}x^{*3} + 2c_{10}c_{20}c_{30}c_{50}x^{*2}y^* \\ &\quad + 3c_{20}^2c_{30}\,c_{50}x^*y^{*2} - c_{40}c_{50}, \\
b_3 &= -2c_{10}c_{30}c_{50}y^* + 6c_{10}^2c_{30}c_{50}x^*y^* + 2c_{10}c_{20}c_{30}c_{50}y^{*2}, \\
b_4 &= c_{30}c_{50} - 2c_{10}c_{30}c_{50}x^* - 2c_{20}c_{30}c_{50}y^* + 3c_{10}^2c_{30}c_{50}x^{*2} + 4c_{10}c_{20}c_{30}c_{50}x^*y^* \\ &\quad + 3c_{20}^2c_{30}c_{50}y^{*2}, \\
b_5 &= -2c_{20}c_{30}c_{50}x^* + 2c_{10}c_{20}c_{30}c_{50}x^{*2} + 6c_{20}^2c_{30}c_{50}x^*y^*, \\
b_6 &= c_{30}x^*y^* - c_{10}c_{30}x^{*2}y^* - c_{20}c_{30}x^*y^{*2} + c_{10}^2c_{30}x^{*3}y^* + c_{10}c_{20}\,c_{30}x^{*2}y^{*2} + c_{20}^2c_{30}x^*y^{*3}, \\
b_7 &= c_{30}y^* - 2c_{10}c_{30}x^*y^* - c_{20}c_{30}y^{*2} + 3c_{10}^2c_{30}x^{*2}y^* + 2c_{10}c_{20}c_{30}x^*y^{*2} + c_{20}^2c_{30}y^{*3}, \\
b_8 &= c_{30}x^* - c_{10}c_{30}x^{*2} - 2c_{20}c_{30}x^*y^* + c_{10}^2c_{30}x^{*3} + 2c_{10}c_{20}c_{30}x^{*2}y^* + 3c_{20}^2c_{30}x^*y^{*2} - c_{40}, \\
b_9 &= -2c_{10}c_{30}y^* + 6c_{10}^2c_{30}x^*y^* + 2c_{10}c_{20}c_{30}y^{*2}, \\
b_{10} &= c_{30} - 2c_{10}c_{30}x^* - 2c_{20}c_{30}y^* + 3c_{10}^2c_{30}x^{*2} + 4c_{10}c_{20}c_{30}x^*y^* + 3c_{20}^2c_{30}y^{*2}, \\
b_{01} &= -2c_{20}c_{30}x^* + 2c_{10}c_{20}c_{30}x^{*2} + 6c_{20}^2c_{30}x^*y^*,
\end{aligned}$$

Using the following transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} a_2 & a_2 \\ -1 - a_1 & \lambda_2 - a_1 \end{pmatrix}\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$$

where T ia an invertible matrix, can be written model (11) we have

$$\begin{cases} \tilde{x}_{n+1} = -\tilde{x}_n + \phi_1(\tilde{x}_n, \tilde{y}_n, c_5^*) + O(|\tilde{x}_n| + |\tilde{y}_n| + |c_5^*|)^3, \\ \tilde{y}_{n+1} = \lambda_2\tilde{y}_n + \psi_1(\tilde{x}_n, \tilde{y}_n, c_5^*) + O(|\tilde{x}_n| + |\tilde{y}_n| + |c_5^*|)^3, \end{cases} \tag{12}$$

where

$$\phi_1(\tilde{x}_n,\tilde{y}_n,c_5^*)=\frac{1}{a_2(\lambda_2+1)}\{a_2^2(a_3(\lambda_2-a_1)-a_2b_3)-a_2(1+a_1)(a_4(\lambda_2-a_1)-a_2b_4)$$
$$+(1+a_1)^2(a_5(\lambda_2-a_1)-a_2b_5)\}\tilde{x}_n^2$$
$$+\frac{1}{a_2(\lambda_2+1)}\{2a_2(a_3(\lambda_2-a_1)-a_2b_3)+a_2(\lambda_2-a_1)(a_4(\lambda_2-a_1)-a_2b_4)$$
$$-a_2(1+a_1)(a_4(\lambda_2-a_1)-a_2b_4)$$
$$-2(1+a_1)(\lambda_2-a_1)(a_5(\lambda_2-a_1)-a_2b_5)\}\tilde{x}_n\tilde{y}_n$$
$$+\frac{1}{a_2(\lambda_2+1)}\{a_2^2(a_3(\lambda_2-a_1)-a_2b_3)+a_2(\lambda_2-a_1)(a_4(\lambda_2-a_1)-a_2b_4)$$
$$+(\lambda_2-a_1)^2(a_5(\lambda_2-a_1)-a_2b_5)\}\tilde{y}_n^2+\frac{a_6(\lambda_2-a_1)-a_2b_6}{a_2(\lambda_2+1)}c_5^2$$
$$+\frac{a_2(a_7(\lambda_2-a_1)-a_2b_7)-(1+a_1)(a_8(\lambda_2-a_1)-a_2b_8)}{a_2(\lambda_2+1)}\tilde{x}_nc_5^*$$
$$+\frac{a_2(a_7(\lambda_2-a_1)-a_2b_7)-(\lambda_2-a_1)(a_8(\lambda_2-a_1)-a_2b_8)}{a_2(\lambda_2+1)}\tilde{y}_nc_5^*.$$

$$\psi_1(\tilde{x}_n,\tilde{y}_n,c_5^*)=\frac{1}{a_2(\lambda_2+1)}\{a_2^2(a_3(1+a_1)+a_2b_3)-a_2(1+a_1)(a_4(1+a_1)+a_2b_4)$$
$$+(1+a_1)^2(a_5(1+a_1)+a_2b_5)\}\tilde{x}_n^2$$
$$+\frac{1}{a_2(\lambda_2+1)}\{2a_2(a_3(1+a_1)+a_2b_3)+a_2(\lambda_2-a_1)(a_4(1+a_1)+a_2b_4)$$
$$-a_2(1+a_1)(a_4(1+a_1)+a_2b_4)+2(1+a_1)(\lambda_2-a_1)(a_5(1+a_1)+a_2b_5)\}\tilde{x}_n\tilde{y}_n$$
$$+\frac{1}{a_2(\lambda_2+1)}\{a_2^2(a_3(1+a_1)+a_2b_3)+a_2(\lambda_2-a_1)(a_4(1+a_1)+a_2b_4)$$
$$+(\lambda_2-a_1)^2(a_5(1+a_1)+a_2b_5)\}\tilde{y}_n^2+\frac{a_6(1+a_1)-a_2b_6}{a_2(\lambda_2+1)}c_5^2$$
$$+\frac{a_2(a_7(1+a_1)+a_2b_7)-(1+a_1)(a_8(1+a_1)+a_2b_8)}{a_2(\lambda_2+1)}\tilde{x}_nc_5^*$$
$$+\frac{a_2(a_7(1+a_1)+a_2b_7)+(\lambda_2-a_1)(a_8(1+a_1)+a_2b_8)}{a_2(\lambda_2+1)}\tilde{y}_nc_5^*$$

From the center manifold theorem given by [14] and [19], we determine center manifold $W^c(0,0)$ of model (12) at fixed point (0,0) in a small neighborhood of c_5^* which can be approximately defined as follows:

$$W^c(0.0)=\{(\tilde{x}_n,\tilde{y}_n):\tilde{y}_n=\tilde{a}_0c_5^*+\tilde{a}_1\tilde{x}_n^2+\tilde{a}_2\tilde{x}_nc_5^*+\tilde{a}_3c_5^{*^2}+O(|\tilde{x}_n|+|c_5^*|)^3\},$$

where

$$\tilde{a}_0=\frac{a_6(1+a_1)-a_2b_6}{a_2(1-\lambda_2^2)},$$
$$\tilde{a}_1=\frac{(1+a_1)(a_4(1+a_1)+a_2b_4)}{\lambda_2^2-1}+\frac{a_3(1+a_1)+a_2b_3}{1-\lambda_2^2}+\frac{(1+a_1)^2(a_5(1+a_1)+a_2b_5)}{a_2(1-\lambda_2^2)},$$
$$\tilde{a}_2=\frac{(1+a_1)(a_8(1+a_1)+a_2b_8)}{a_2(\lambda_2+1)^2}+\frac{a_7(1+a_1)+a_2b_7}{a_2(\lambda_2+1)^2}+\frac{2\tilde{a}_1(a_6(\lambda_2-a_1)-a_2b_6)}{a_2(\lambda_2+1)^2}$$
$$\tilde{a}_3=\frac{a_6(\lambda_2-a_1)-a_2b_6}{a_2(\lambda_2^2-1)}\left[\frac{\tilde{a}_1(a_6(\lambda_2-a_1)-a_2b_6)}{a_2(\lambda_2+1)}+\tilde{a}_2\right].$$

From model (12) we obtain that, which is restricted center manifold $W^c(0,0)$, we have

$$H(\tilde{x}_n)=-\tilde{x}_n+\phi(\tilde{x}_n,\tilde{y}_n,c_5^*)$$
$$=-\tilde{x}_n+h_1\tilde{x}_n^2+h_2\tilde{x}_nc_5^*+h_3c_5^{*^2}+h_4\tilde{x}_n^2c_5^*+h_5\tilde{x}_nc_5^{*^2}+h_6\tilde{x}_n^3+h_7c_5^{*^3}+O(|\tilde{x}_n|+|c_5^*|)\qquad(13)$$

where

$$h_1 = \frac{a_2(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{(1 + a_1)(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} + \frac{(1 + a_1)^2(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)},$$

$$h_2 = \frac{2\tilde{a}_0(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{a_0((\lambda_2 - a_1) - (1 + a + 1))(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} - \frac{2\tilde{a}_0(1 + a_1)(\lambda_2 - a_1)(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)} - \frac{(1 + a_1)(a_8(\lambda_2 - a_1) - a_8 b_8)}{a_2(\lambda_2 + 1)} + \frac{a_7(\lambda_2 - a_1) - a_2 b_7}{\lambda_2 + 1},$$

$$h_3 = \frac{a_2\tilde{a}_0^2(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{\tilde{a}_0^2(\lambda_2 - a_1)(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} + \frac{\tilde{a}_0(a_7(\lambda_2 - a_1) - a_2 b_7)}{\lambda_2 + 1} + \frac{\tilde{a}_0^2(\lambda_2 - a_1)^2(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)} + \frac{\tilde{a}_0(\lambda_2 - a_1)(a_8(\lambda_2 - a_1) - a_2 b_8)}{a_2(\lambda_2 + 1)},$$

$$h_4 = \frac{2\tilde{a}_2(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{2\tilde{a}_0\tilde{a}_1 a_2(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{\tilde{a}_1(a_7(\lambda_2 - a_1) - a_2 b - 7)}{\lambda_2 + 1} + \frac{\tilde{a}_2((\lambda_2 - a_1) - (1 + a_1))(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} + \frac{2\tilde{a}_0\tilde{a}_1(\lambda_2 - a_1)^2(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)} + \frac{2\tilde{a}_0\tilde{a}_1(\lambda_2 - a_1)(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} - \frac{2\tilde{a}_2(1 + a_1)(\lambda_2 - a_1)(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)} + \frac{\tilde{a}_1(\lambda_2 - a_1)(a_8(\lambda_2 - a_1) - a_2 b_8)}{a_2(\lambda_2 + 1)},$$

$$h_5 = \frac{2\tilde{a}_3(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{\tilde{a}_3((\lambda_2 - a_1) - (1 + a_1))(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} + \frac{\tilde{a}_2(a_7(\lambda_2 - a_1) - a_2 b_7)}{\lambda_2 + 1} - \frac{2\tilde{a}_3(1 + a_1)(\lambda_2 - a_1)(a_5(\lambda_2 - a - 1) - a_2 b_5)}{a_2(\lambda_2 + 1)} + \frac{\tilde{a}_2(\lambda_2 - a_1)(a_8(\lambda_2 - a_1) - a_2 b_8)}{a_2(\lambda_2 + 1)},$$

$$h_6 = \frac{2\tilde{a}_1(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{\tilde{a}_1((\lambda_2 - a_1) - (1 - a_1))(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} - \frac{2\tilde{a}_1(1 + a_1)(\lambda_2 - a_1)(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)},$$

$$h_7 = \frac{2\tilde{a}_0\tilde{a}_3(\lambda_2 - a_1)^2(a_5(\lambda_2 - a_1) - a_2 b_5)}{a_2(\lambda_2 + 1)} + \frac{2\tilde{a}_0\tilde{a}_3(\lambda_2 - a_1)(a_4(\lambda_2 - a_1) - a_2 b_4)}{\lambda_2 + 1} + \frac{2\tilde{a}_0\tilde{a}_3 a_2(a_3(\lambda_2 - a_1) - a_2 b_3)}{\lambda_2 + 1} + \frac{\tilde{a}_3(a_7(\lambda_2 - a_1) - a_2 b_7)}{\lambda_2 + 1} + \frac{\tilde{a}_3(\lambda_2 - a_1(a_8(\lambda_2 - a_1) - a_2 b_8))}{a_2(\lambda_2 + 1)}.$$

From map (13) a flip bifurcation, we require that two discriminatory quantities β_1 and β_2 are not zero,

$$\frac{\partial H}{\partial \tilde{x}_n} = -1 + 2h_1\tilde{x}_n + h_2 c_5^* + 2h_4\tilde{x}_n c_5^* + h_5 c_5^* + 3h_6\tilde{x}_n^2,$$

$$\frac{\partial^2 H}{\partial \tilde{x}_n \partial c_5^*} = h_2 + 2h_4\tilde{x}_n + 2h_5 c_5^*,$$

$$\frac{\partial^2 H}{\partial \tilde{x}_n^2} = 2h_1 + 2h_4 c_5^* + 6h_6\tilde{x}_n,$$

$$\frac{\partial H}{\partial c_5^*} = h_2\tilde{x}_n + 2h_3 c_5^* + h_4\tilde{x}_n^2 + 2h_5\tilde{x}_n c_5^* + 3h_7 c_5^*,$$

$$\frac{\partial^3 H}{\partial \tilde{x}_n^3} = 6h_6,$$

$$\beta_1 = \left(\frac{\partial^2 H}{\partial \tilde{x}_n \partial c_5^*} + \frac{1}{2}\frac{\partial H}{\partial c_5^*}\frac{\partial^2 H}{\partial \tilde{x}_n^2}\right)|_{(0,0)} = h_2,$$

$$\beta_2 = \left(\frac{1}{6}\frac{\partial^3 H}{\partial \tilde{x}_n^3} + \left(\frac{1}{2}\frac{\partial^2 H}{\partial \tilde{x}_n^2}\right)^2\right)|_{(0,0)} = h_1^2 + h_6,$$

Finally, as a result of the above analysis and theorem 3.1 in [24] we have the following.

Theorem 3.1 β_1 and β_2 then the model (3) undergoes a flip bifurcation at fixed point $E_2(x^*, y^*)$ where the parameter c_5 varies in a small neighborhood of c_5^*. Moreover, if $\beta_2 > 0(resp.\ \beta_2 < 0)$, then the periodic-2 point that bifurcation from $E_2(x^*, y^*)$ are stable (resp. unstable).

And condition (iii) of Lemma (2.7), we can get that the eigenvalue $\lambda_{1,2}$ of $J(E_2)$ are pair of conjugate complex numbere. Whene the parameter of model (3) are located in the following set

$$A_3 = \{(c_1, c_2, c_3, c_4, c_5) \in (0, +\infty) : c_5 M^* = N^*, \quad N^* < 2\sqrt{M^*}\}.$$

We will discuss the Hopf bifurcation of the fixed $E_2(x^*, y^*)$ if all parameter of model (3) vary in a small neighborhood of A_3.

From Lemma 2.7, has shown that $E_2(x^*, y^*)$ is non-hyperbolic whene $c_5 M^* = N^*$. To study Hopf bifurcation of model (3) at fixed point $E_2(x^*, y^*)$ in a small neighborhood of A_3. Taking parameter $(c_1, c_2, c_3, c_4, c_5) = (c_{11}, c_{21}, c_{31}, c_{41}, c_{51}) \in A_3$,

$$\begin{cases} x_{n+1} = x_n + c_{51} x_n \left(1 - x_n - \dfrac{y_n}{(1 + c_{11} x_n)(1 + c_{21} y_n)}\right) \\ y_{n+1} = y_n + c_{51} y_n \left(\dfrac{c_{31} x_n}{(1 + c_{11} x_n)(1 + c_{21} y_n)} - c_{41}\right) \end{cases} \quad (14)$$

Give a perturbation $\bar{c}_5$ at c_{51}, the model (14) is defined by

$$\begin{cases} x_{n+1} = x_n + (c_{51} + \bar{c}_5) x_n \left(1 - x_n - \dfrac{y_n}{(1 + c_{11} x_n)(1 + c_{21} y_n)}\right) \\ y_{n+1} = y_n + (c_{51} + \bar{c}_5) y_n \left(\dfrac{c_{31} x_n}{(1 + c_{11} x_n)(1 + c_{21} y_n)} - c_{41}\right) \end{cases} \quad (15)$$

Moving $E_2(x^*, y^*)$ to the origin, let $u_n = x_n - x^*$, $v_n = y_n - y^*$. Furthere expanding Eq. (15) a Taylor series at $(u_n, v_n) = (0,0)$ to the second order, have

$$\begin{cases} u_{n+1} = a_{11} u_n + a_{21} v_n + a_{31} u_n^2 + a_{41} u_n v_n + a_{51} v_n^2 + a_{61} u_n^2 v_n + a_{71} u_n v_n^2 + O(|u_n| + |v_n|),^3 \\ v_{n+1} = b_{11} u_n + b_{21} v_n + b_{31} u_n^2 + b_{41} u_n v_n + b_{51} v_n^2 + b_{61} u_n^2 v_n + b_{71} u_n v_n^2 + O(|u_n| + |v_n|)^3, \end{cases} \quad (16)$$

where

$$\begin{aligned}
a_{11} &= 1 + c_{51} - 2c_{51}x^* - c_{51}y^* + 2c_{11}c_{51}x^*y^* + c_{21}c_{51}y^{*2} - 3c_{11}^2c_{51}x^{*^2}y^* - 2c_{11}c_{21}c_{51}x^{*^2}y^{*^2} \\ &\quad - c_{21}^2c_{51}y^{*^3}, \\
a_{21} &= -c_{51}x^* + c_{11}c_{51}x^{*^2} + 2c_{21}c_{51}x^*y^* - c_{11}c_{51}x^{*^3} - 2c_{11}c_{21}c_{51}x^{*2}y^* - 3c_{21}^2c_{51}x^*y^{*^2}, \\
a_{31} &= -2c_{51} + 2c_{11}c_{51}y^* - 6c_{11}^2c_{51}x^*y^* - 2c_{11}c_{21}c_{51}y^{*^2}, \\
a_{41} &= -c_{51} + 2c_{11}c_{51}x^* + 2c_{21}c_{51}y^* - 3c_{11}^2c_{51}x^* - 4c_{11}c_{21}c_{51}x^*y^* - 3c_{21}^2c_{51}y^{*^2}, \\
a_{51} &= 2c_{21}c_{51}x^* - 2c_{11}c_{21}c_{51}x^{*2} - 6c_{21}^2c_{51}x^*y^*, \\
a_{61} &= 2c_{11}c_{51} - 6c_{11}^2c_{51}x^* - 4c_{11}c_{21}c_{51}y^*, \\
a_{71} &= 2c_{21}c_{51} - 4c_{11}c_{21}c_{51}x^* - 6c_{21}^2c_{51}y^*, \\
b_{11} &= c_{31}c_{51}y^* - 2c_{11}c_{31}c_{51}x^*y^* - c_{21}c_{31}c_{51}y^{*2} + 3c_{11}^2c_{31}c_{51}x^{*^2}y^* + 2c_{11}c_{21}c_{31}c_{51}x^*y^{*^2} \\ &\quad + c_{21}^2c_{31}c_{51}y^{*^3}, \\
b_{21} &= 1 + c_{31}c_{51}x^* - c_{11}c_{31}c_{51}x^{*^2} - 2c_{21}c_{31}c_{51}x^*y^* + c_{11}^2c_{31}c_{51}x^{*3} + 2c_{11}c_{21}c_{31}c_{51}x^{*^2}y^* \\ &\quad + 3c_{21}^2c_{31}c_{51}x^*y^{*^2} - c_{41}c_{51}, \\
b_{31} &= -2c_{11}c_{31}c_{51}y^* + 6c_{11}^2c_{31}c_{51}x^*y^* + 2c_{11}c_{21}c_{31}c_{51}y^{*^2}, \\
b_{41} &= c_{31}c_{51} - 2c_{11}c_{31}c_{51}x^* - 2c_{21}c_{31}c_{51}y^* + 3c_{11}^2c_{31}c_{51}x^{*^2} + 4c_{11}c_{21}c_{31}c_{51}x^*y^* \\ &\quad + 3c_{21}^2c_{31}c_{51}y^{*^2}, \\
b_{51} &= -2c_{21}c_{31}c_{51}x^* + 2c_{11}c_{21}c_{31}c_{51}x^{*^2} + 6c_{21}^2c_{31}c_{51}x^*y^*, \\
b_{61} &= -2c_{11}c_{31}c_{51} + 6c_{11}^2c_{31}c_{51}x^* + 4c_{11}c_{21}c_{31}c_{51}y^*, \\
b_{71} &= -2c_{21}c_{31}c_{51} + 4c_{11}c_{21}c_{31}c_{51}x^* + 6c_{21}^2c_{31}c_{51}y^*,
\end{aligned}$$

The characteristic equation associated with the linearization of Eq. (16) at $(u_n, v_n) = (0,0)$ is

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$$\lambda^2 + p(\bar{c}_5)\lambda + q(\bar{c}_5) = 0 \qquad (17)$$

where

$$p(\bar{c}_5) = -2 + (\bar{c}_5 + c_{51})N^*$$
$$q(\bar{c}_5) = 1 - (\bar{c}_5 + c_{51})N^* + (\bar{c}_5 + c_{51})^2 M^*$$

whene $\bar{c}_5 = 0$, λ_1,λ_2 are roots complex conjugate of characteristic equation (17) are

$$\lambda_{1,2} = \frac{p(\bar{c}_5) \pm i\sqrt{4q(\bar{c}_5) - p^2(\bar{c}_5)}}{2},$$

Since $|\lambda_{1,2}(\bar{c}_5)| = \sqrt{q(\bar{c}_5)}$. From $|\lambda_{1,2}(\bar{c}_5)| = 1$, whene $\bar{c}_5 = 0$ we have $q(0) = 1 - N^* c_{51} + M^* c_{51}^2 = 1$.

Hence $c_{51}M^* = N^*$. Therefore

$$\frac{d|\lambda_{1,2}(\bar{c}_5)|}{dc_{\bar{c}_5}}\Big|_{\bar{c}_5=0} = -\frac{N^*}{2} \neq 0.$$

Furthermore, the condition $\lambda_{1,2}^n \neq 1$ $(n = 1,2,3,4)$ imply $p(0) = -2,0,1,2$. Not that $N^* < 2\sqrt{M^*}$ means $(p(0))^2 < 4$ and then $p(0) \neq -2,2$. We show that $p(0) \neq 0,1$ and we get $p(0) = -2 + N^* c_{51} \neq 0,1$. Then $N^* c_{51} \neq 2,3$ and thus

$$c_{51}x^* + c_{51}(1 - x^*)\left(c_{21}c_{41}(1 + c_{11}x^*) - \frac{c_{11}x^*}{1 + c_{11}x^*}\right) \neq 2,3 \quad (18)$$

Therefore, whene $\bar{c}_5 = 0$ and (18) hold, the eigenvalue $\lambda_{1,2}$ do not lie at the point where the unit circle intersect the coordinate axes.

Let

$$\alpha = Re(\lambda_{1,2}) = 1 - \frac{1}{2}c_{51}\left[x^* + (1 - x^*)\left(c_{21}c_{41}(1 + c_{11}x^*) - \frac{c_{11}x^*}{1 + c_{11}x^*}\right)\right],$$
$$\beta = Im(\lambda_{1,2}) = \frac{1}{2}c_{51}\sqrt{4M^* - N^{*2}},$$

Using the translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$$

where T ia an invertible matrix, can be written Eq. (16) we have

$$\begin{cases} \tilde{x}_{n+1} = \alpha\tilde{x}_n - \beta\tilde{y}_n + \phi_2(\tilde{x}_n, \tilde{y}_n), \\ \tilde{y}_{n+1} = \beta\tilde{x}_n + \alpha\tilde{y}_n + \psi_2(\tilde{x}_n, \tilde{y}_n), \end{cases} \qquad (19)$$

where

$$\phi_2(\tilde{x}_n, \tilde{y}_n) = \frac{1}{\beta}[(b_{31} - \alpha a_{31})\tilde{y}_n^2 + (b_{41} - \alpha a_{41})\tilde{y}_n(\beta\tilde{x}_n + \alpha\tilde{y}_n) + (b_{51} - \alpha a_{51})(\beta\tilde{x}_n + \alpha\tilde{y}_n)^2$$
$$+ (b_{61} - \alpha a_{61})\tilde{y}_n^2(\beta\tilde{x}_n + \alpha\tilde{y}_n) + (b_{71} - \alpha a_{71})\tilde{y}_n(\beta\tilde{x}_n + \alpha\tilde{y}_n)^2] + O(|\tilde{x}_n| + |\tilde{y}_n|)^4,$$
$$\psi_2(\tilde{x}_n, \tilde{y}_n) = a_{31}\tilde{y}_n^2 + a_{41}\tilde{y}_n(\beta\tilde{x}_n + \alpha\tilde{y}_n) + a_{51}(\beta\tilde{x}_n + \alpha\tilde{y}_n)^2 + a_{61}\tilde{y}_n^2(\beta\tilde{x}_n + \alpha\tilde{y}_n)$$
$$+ a_{71}\tilde{y}_n(\beta\tilde{x}_n + \alpha\tilde{y}_n)^2 + O(|\tilde{x}_n| + |\tilde{y}_n|)^4,$$

and

$$\phi_{2\tilde{x}\tilde{x}} = 2\beta(b_{51} - \alpha a_{51}),\ \phi_{2\tilde{x}\tilde{y}} = (b_{41} - \alpha a_{41}) + 2\alpha(b_{51} - \alpha a_{51}),\ \phi_{2\tilde{x}\tilde{x}\tilde{y}} = 2\beta(b_{71} - \alpha a_{71}),$$
$$\phi_{2\tilde{y}\tilde{y}} = \frac{1}{\beta}[2(b_{31} - \alpha a_{31}) + 2\alpha(b_{41} - \alpha a_{41}) + 2\alpha^2(b_{51} - \alpha a_{51})],\ \phi_{2\tilde{x}\tilde{x}\tilde{x}} = 0,$$
$$\phi_{2\tilde{x}\tilde{y}\tilde{y}} = 2(b_{61} - \alpha a_{61}) + 4(b_{71} - \alpha a_{71}),\ \phi_{2\tilde{y}\tilde{y}\tilde{y}} = \frac{1}{\beta}[6\alpha(b_{61} - \alpha a_{61}) + 6\alpha^2(b_{71} - \alpha a_{71})],$$
$$\psi_{2\tilde{x}\tilde{x}} = \beta^2 a_{51},\ \psi_{2\tilde{x}\tilde{y}} = \beta a_{41} + 2\alpha\beta a_{51},\ \psi_{2\tilde{y}\tilde{y}} = 2a_{31} + 2\alpha a_{41} + 2\alpha^2 a_{51},\ \psi_{2\tilde{x}\tilde{x}\tilde{x}} = 0,$$
$$\psi_{2\tilde{x}\tilde{y}\tilde{y}} = 2\beta a_{51} + 4\alpha\beta a_{71},\ \psi_{2\tilde{x}\tilde{x}\tilde{y}} = 2\beta^2 a_{71},\ \psi_{2\tilde{y}\tilde{y}\tilde{y}} = 6\alpha a_{61} + 6\alpha^2 a_{71},$$

The quantity a^* is not be zero [24], in order for the Eq. (19) to undergo Hopf bifurcation.

$$a^* = Re\left[\frac{(1 - 2\lambda)\bar{\lambda}^2}{1 - \lambda}\xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}),$$

where

$$\xi_{20} = \frac{1}{8}[\phi_{2\tilde{x}\tilde{x}} - \phi_{2\tilde{y}\tilde{y}} + 2\psi_{2\tilde{x}\tilde{y}} + i(\psi_{2\tilde{x}\tilde{x}} - \psi_{2\tilde{y}\tilde{y}} - 2\phi_{2\tilde{x}\tilde{y}})],$$
$$\xi_{02} = \frac{1}{8}[\phi_{2\tilde{x}\tilde{x}} - \phi_{2\tilde{y}\tilde{y}} + 2\psi_{2\tilde{x}\tilde{y}} + i(\psi_{2\tilde{x}\tilde{x}} - \psi_{2\tilde{y}\tilde{y}} + 2\phi_{2\tilde{x}\tilde{y}})],$$
$$\xi_{11} = \frac{1}{4}[\phi_{2\tilde{x}\tilde{x}} + \phi_{2\tilde{y}\tilde{y}} + i(\psi_{2\tilde{x}\tilde{x}} + 2\phi_{2\tilde{x}\tilde{y}})],$$
$$\xi_{21} = \frac{1}{16}[\phi_{2\tilde{x}\tilde{x}\tilde{x}} + \phi_{2\tilde{x}\tilde{y}\tilde{y}} + \psi_{2\tilde{x}\tilde{x}\tilde{y}} + i(\psi_{2\tilde{x}\tilde{x}\tilde{x}} - \psi_{2\tilde{x}\tilde{y}\tilde{y}} - \phi_{2\tilde{x}\tilde{x}\tilde{y}} - \psi_{2\tilde{y}\tilde{y}\tilde{y}})],$$

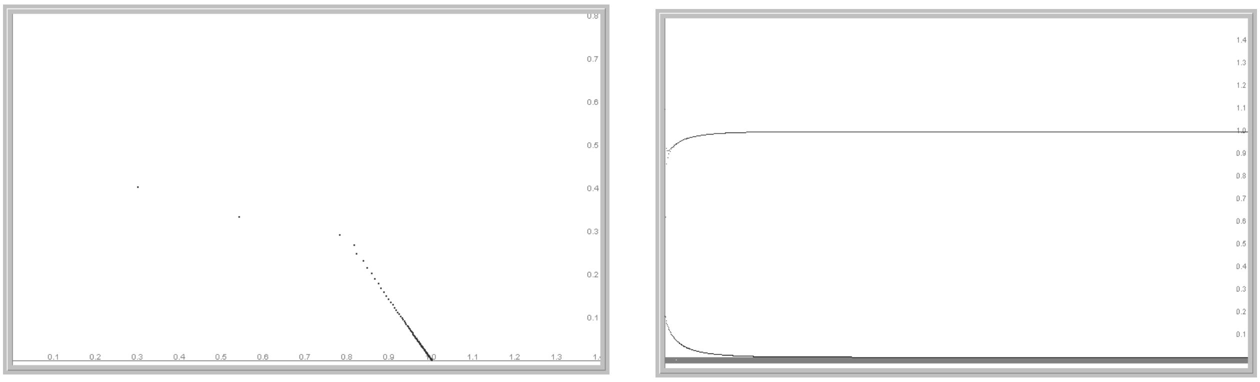
Finally, as a result of the above analysis and theorem 3.5.2 in [24] we obtain the followin result.

Theorem 3.2 Whene the parameter $\bar{c}_5$ change in a small neighborhood of the c_{51}, the model (3) undergo Hopf bifurcation at $E_2(x^*, y^*)$ if the condition (19) hold $a^* = 0$. Furthermore, if $a^* < 0$ (or $a^* > 0$) an attracting (or repelling) invariant closed curve bifurcation at $E_2(x^*, y^*)$.

4. Numerical simulation

In this section, to demonstrate dynamical behavior of the model (3) by giving some numerical evidence and the bifurcation diagram, phase potraites were used to illustrate the above analytical result and explain new dynamical behavior by using numerical simulations. We consider the parameter values, we have the following cases

For case 1, taking parameter $c_1 = 0.35, c_2 = 0.52, c_3 = 0.15, c_4 = 0.32$ and $c_5 = 2$ it shows that in Lemma (2.4) $E_1(1,0)$ is non-hyperbolic for $c_5 = 2$. From Figure 1, is verifies Proposition (2.5), whene $c_5 = 2$ with multiplies $\lambda_1 = 1, \lambda_2 = 0.5822222222$ and $(c_1 . c_2, c_3, c_4, c_5) \in \Re$. So $E_1(1,0)$ which is attracting.



(a) (b)

Figure 1: The diagram for fixed point E_1 of model (3) with intial value is (0.9,0.7). (a) attracting whene $c_5 = 2$, (b) time series of (a).

For case 2, taking parameter $c_1 = 0.35, c_2 = 0.52, c_3 = 0.15$ and $c_4 = 0.12$ from Lemma (2.4), we see that stability fixed point $E_1(1,0)$ of the model (3) happenes $c_5 < 2$ and loses it is stability at $c_5 = 2$ and periodic doubling bifurcation for $c_5 > 2$, it will be a chaotic set when increasing of c_5. The phase portraits which are related Figure 2, are given in Figure 3. For $c_5 \in [2,2.56]$, there are periodic-2,4,8 and 16. When $c_5 = 2.57,2.58$
Are chaotic set.

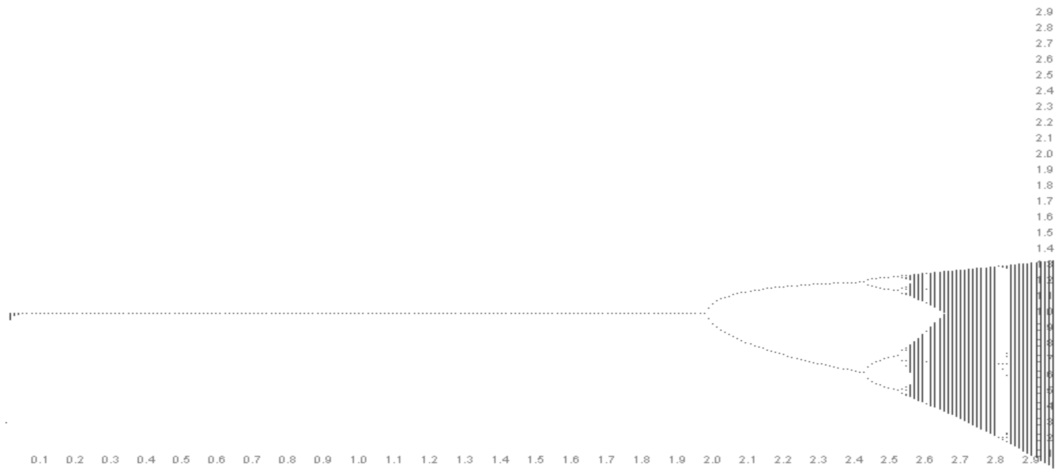


Figure 2: The bifurcation diagram for fixed point E_1 of model (3) with the initial value is (0.9,0.7)

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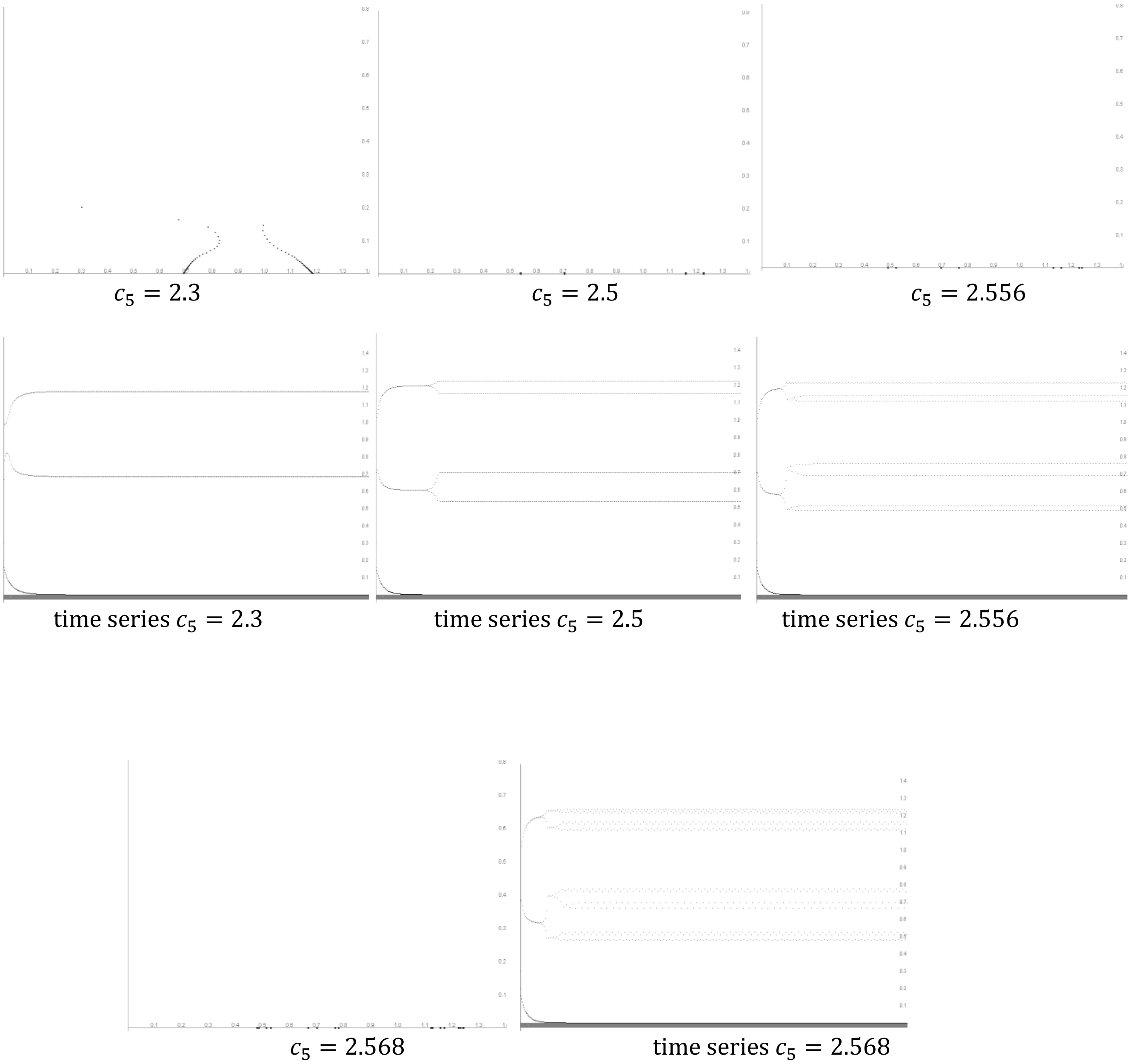
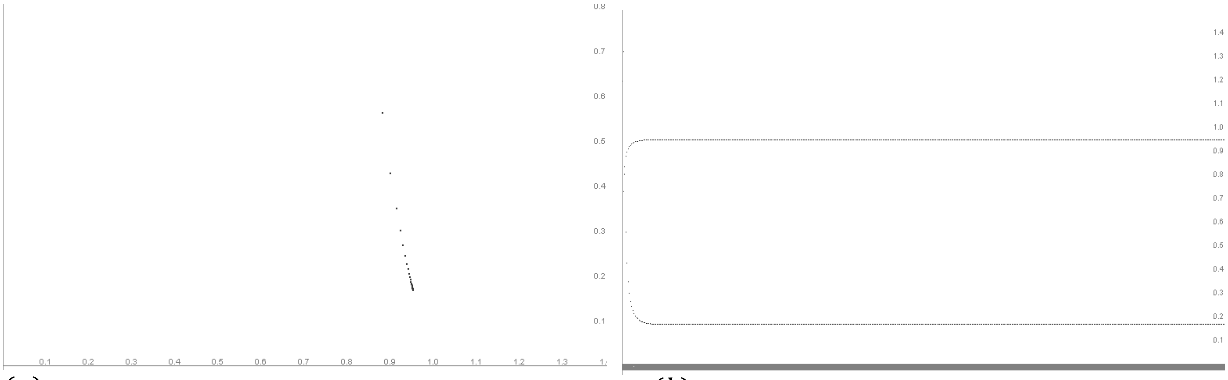


Figure 3: Phase portraits of different values c_5 equivalent to figure 2

For case 3, taking parameter $c_1 = 0.35, c_2 = 0.52, c_3 = 0.15$ and $c_4 = 0.12$, we know the model (3) has only one positive fixed point $E_2(0.9505, 0.16650)$. From Figure 5, if $c_5 = 1.2$ it shows that E_2 is stable.



(a) (b)

Figure 4: The stability for fixed point E_2 with the intial value is (1.2,2.3)

To confirm Lemma (2.6), a flip bifurcation from the fixed point $E_2(0.9505,0.16650)$ occurs at $c_5 = 2.190277138$ with multiplies $\lambda_1 = 0.6844514937, \lambda_2 = -1$ and $(c_1, c_2, c_3, c_4, c_5) = (1.45,2.5,1.63,0.46,2.190277138) \in A_2$.

According to bifurcation diagrams shown in Figure 5, we see that the fixed point E_2 is stable for $c_5 < 2.190277138$ and loses its stability at the flip bifurcation $c_5 = 2.190277138$ and period doubling phenomena lead to chaos for $c_5 > 2.190277138$. The phase portraits which are related Figure 5, are given in Figure 6. For $c_5 \in [2.19027,2.744]$, there are period-2,4,8 and 16. Whene $c_5 = 2.755,2.76$ are chaotic set.

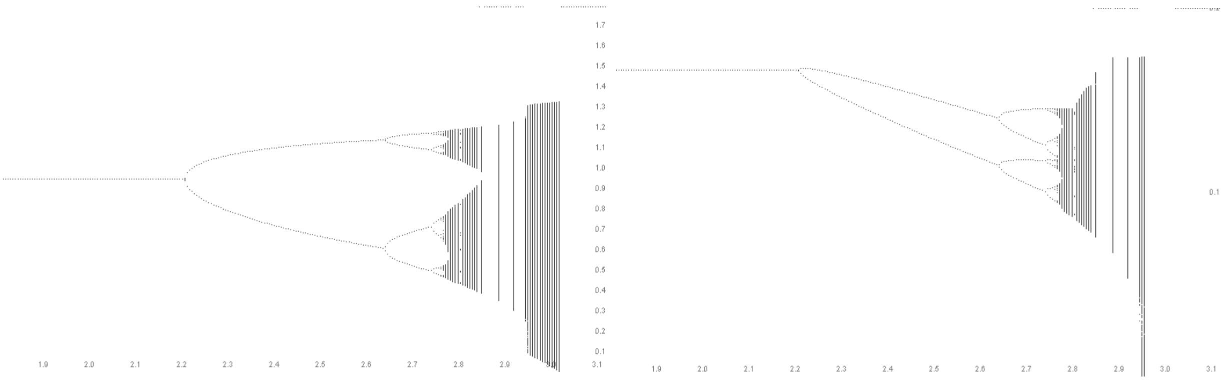


Figure 5: Bifurcation diagram of model (3). (a) Bifurcation for prey, (b) bifurcation for predator with $c_1 = 0.35, c_2 = 0.52, c_3 = 0.15$ and $c_4 = 0.12, c_5 \in [1.9,3.1]$ the initial value is (1.2,2.3).

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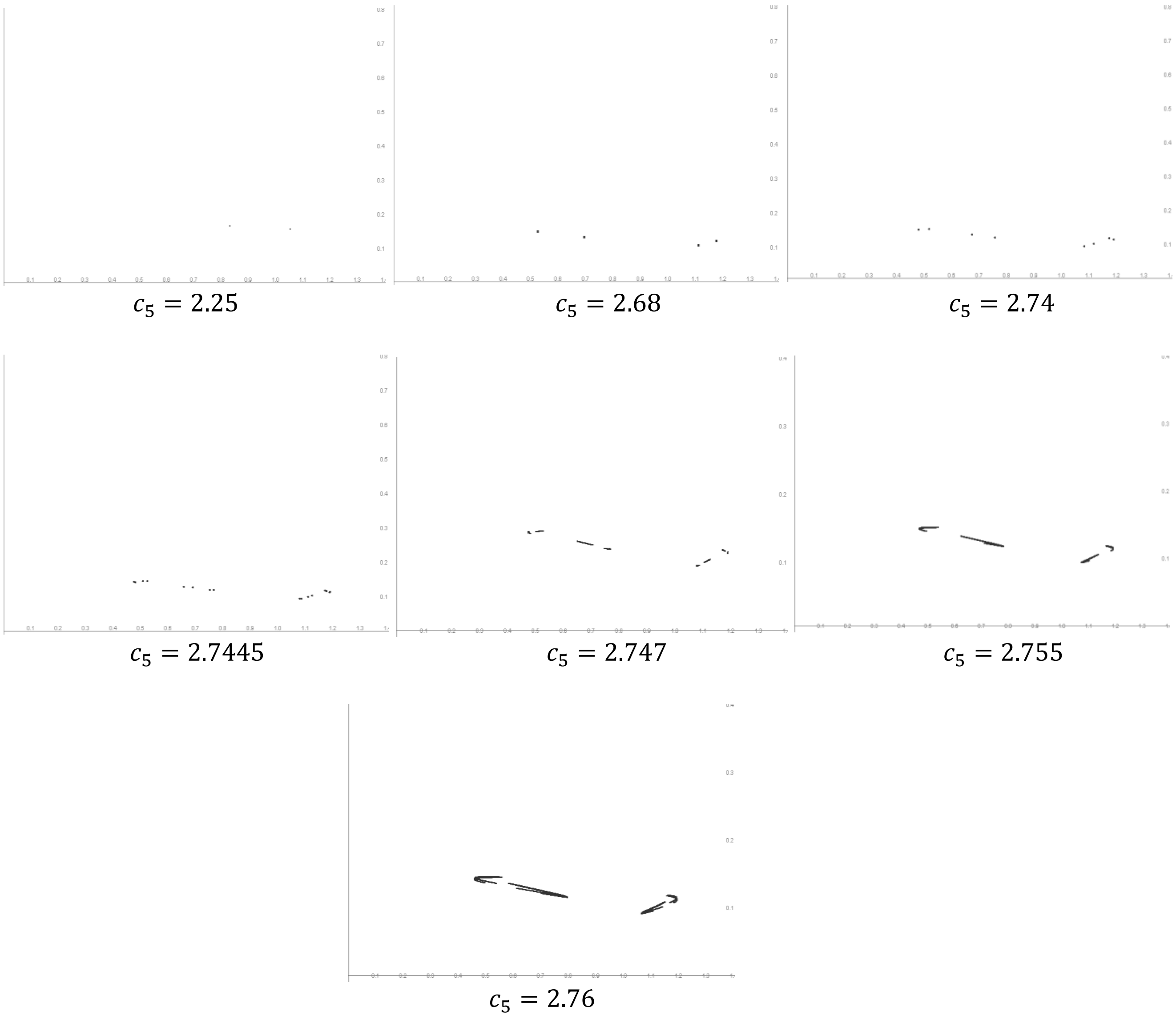

Figure 6: Phase portraits of different values c_5 equivalent to figure 5

For case 4 taking parameter $c_1 = 0.6, c_2 = 0.5, c_3 = 2.2$ and $c_4 = 0.46$ we know the model (3) has only one positive fixed point $E_2(0.40762,0.16650)$. To confirm Lemma (2.7), a Hopf bifurcation appears at fixed point $E_2(0.40762,0.16650)$ for $c_5 = 2.4566$. Also we have $\lambda, \bar{\lambda} = 0.4341614785 \pm 0.9008439802i$ and $(c_1, c_2, c_3, c_4, c_5) = (0.6,0.5,2.2,0.46,2.4566) \in A_3$.

The bifurcation diagram shown in Figuers 7(a) and 7(c) has shown stability fixed point E_2 happens for $c_5 < 2.4566$ and loses its stability at $c_5 = 2.4566$ and an invariant circle appears if $c_5 > 2.4566$. From Figuers 7, there are period-doubling phenomena. Figures 7(b) and 7(d) is local amplification corresponding to (a) and (c).

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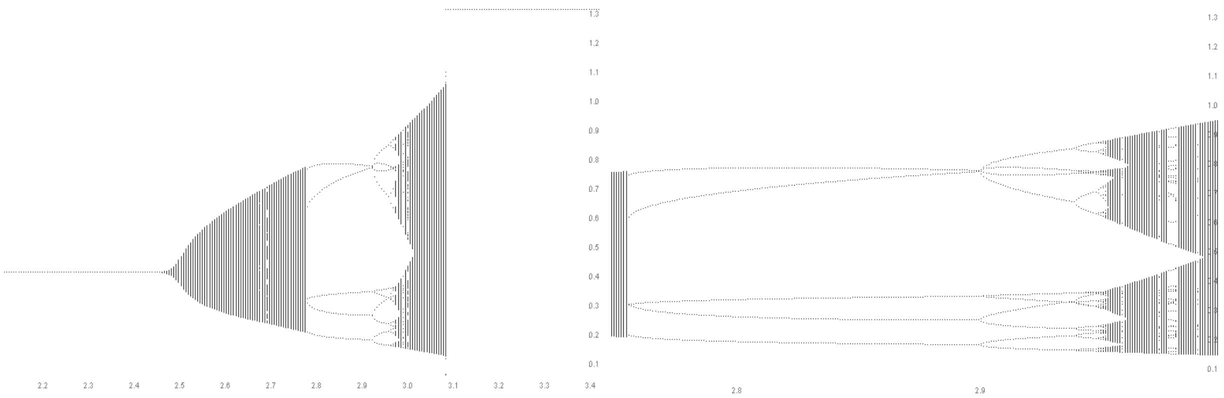
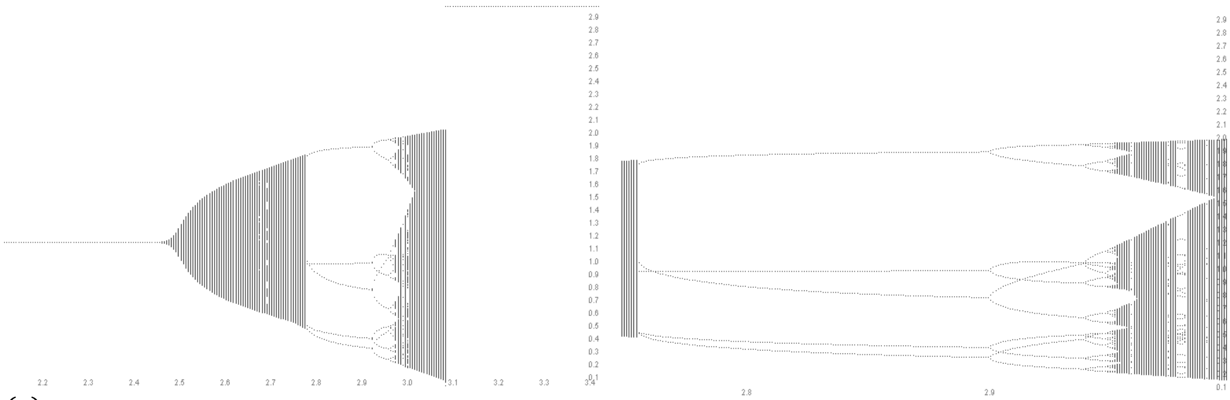
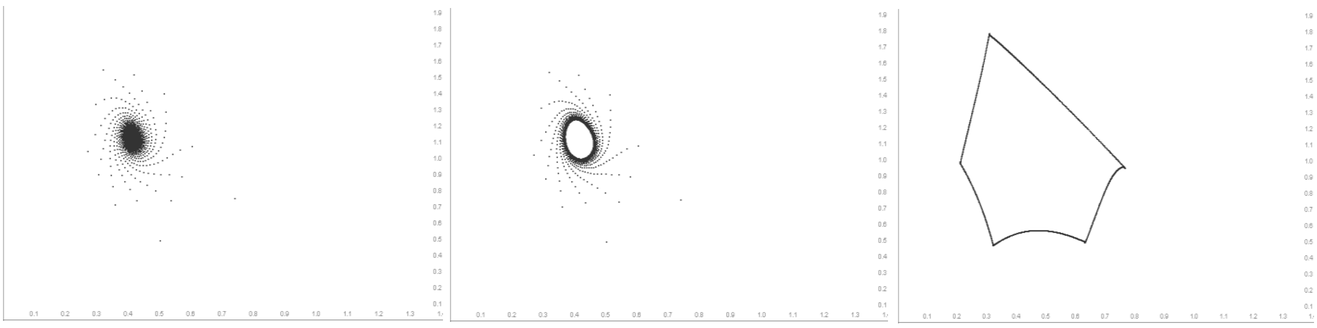


(a) (b)

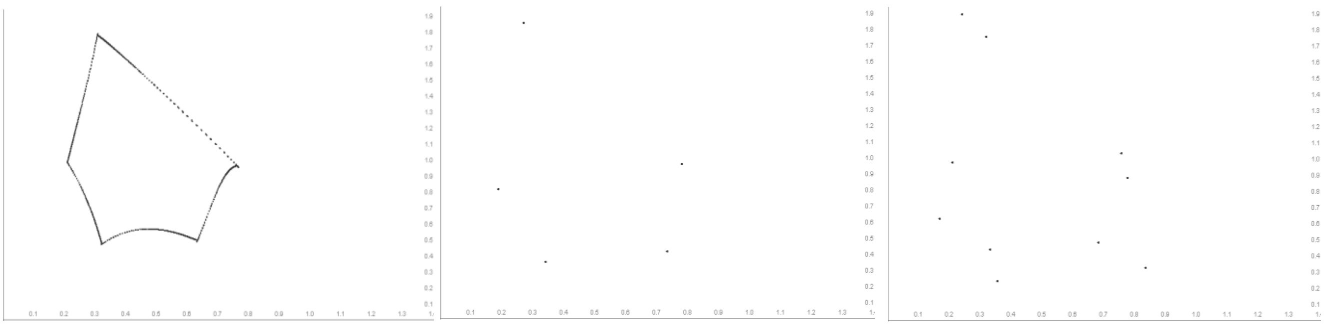


(c) (d)

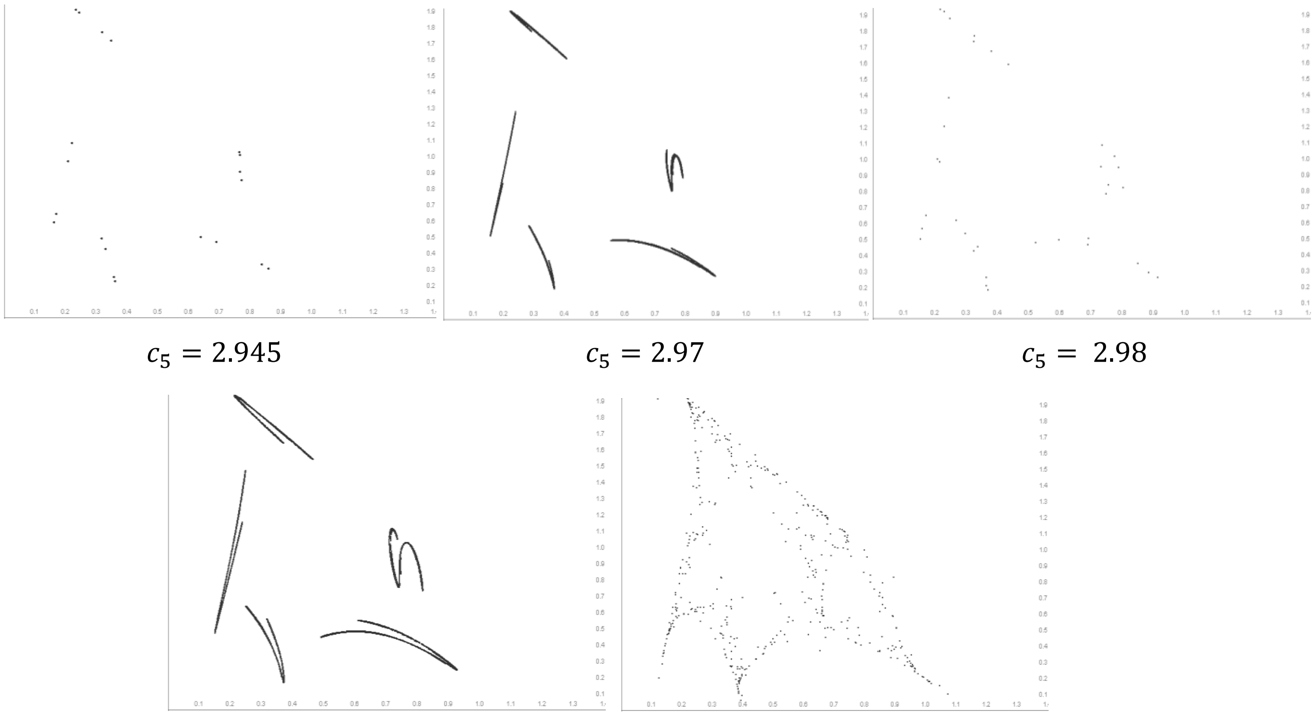
Figure 7: Bifurcation diagram of model (3). (a) Hopf bifurcation for prey, (b) local amplification corresponding to (a). (c) Hopf bifurcation for predator, (d) local amplification corresponding to (c) for $c_5 \in [2.1,3.1]$, the initial value is (0.5,0.5).



$c_5 = 2.475$ $c_5 = 2.485$ $c_5 = 2.756$



$c_5 = 2.7566$ $c_5 = 2.84$ $c_5 = 2.93$



$c_5 = 2.945$ $c_5 = 2.97$ $c_5 = 2.98$

$c_5 = 2.99$ $c_5 = 3.08$

Figure 8: Phase portraits of different values c_5 equivalent to figure 7

The phase portraits which are related Figure 7, are given in Figure 8, for different value c_5 are explained in Figure 8, it illustrate that of smooth invariant circle how it bifurcate from stable fixed point $E_2(0.40762, 0.16650)$. whene $c_5 > 2.4566$ appears a circle curve and radius becomes larger whene c_5 is increases. whene c_5 grows the circle disappears occurs suddenly and period-5,10,20 and 35 orbits appear, for example at $c_5 = 2.48, 2.93$ and it's chaotic whene $c_5 = 2.98, 3.08$.

5 Conclusion

Our goals investigated complex dynamical behavior of model (3) in the closed first quadrant $\mathbb{R}_+^2$. Be sufficient condition for the flip bifucation and Hopf bifurcation at unique positive fixed point by using center manifold theorem and bifurcation theory if c_5 varies of the sets A_1 or A_2 and A_3. Numerical simulation displays unexpected behavior through flip bifurcation which includes of period-2,4,8 and 16 orbits and through a Hopf bifurcation which includes an invariant cycle, of period-5,10,20 and 35 orbits and chaotic sets.

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