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t-modular spaces

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ABSTRACT

In the current study, a new modular type $\rho:(0,\infty)\times X\rightarrow[0,\infty)$ which is called t-modular is defined. Some properties are given and proven, the vector space $X_\rho=\left\{x\in X: \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}}\rho_\alpha(\gamma x)=0\right\}$ is defined, namely *t-modular space* with a norm function on X_ρ being stated.

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t– موديولر

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الملخص

في هذه الورقة البحثية لقد قمنا بتعريف نوع جديد من الموديولر $\rho: 0,\infty) \times X \to [0,\infty)$ وتمت تسميته ب t–موديولر وبرهنا بعض الخصائص وتمكنا أيضا من تعريف فضاء متجاهات على هذا الموديولر $X_\rho = \left\{ x \in X : \lim_{\substack{\gamma\to 0 \\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0 \right\}$ وسمي فضاء t–موديولر وعلى هذا الفضاء عرفنا داله النورم.

1. Introduction

Nakano [1] started researching modulars on linear spaces and the idea of modular linear spaces, which is a generalization of metric spaces, in 1950. Then, it was fully developed by Luxemburg [2], Mazur, Musielak, and Orlicz [3, 4, 5]. Since then, much research has been conducted using the concepts of modulars and modular spaces of different Orlicz spaces [6] and interpolation theory [7, 8]. Although a modular gives fewer features than a norm, it is more logical in many particular circumstances. Remember that [9] provides results on the concept of a partial modular metric space with some fixed points. According to Kowzslowski's formulation [10, 11], a modular on a vector space X is defined as follows:

Definition 1.1: Let X be an arbitrary vector space. A functional $\rho : X \to [0.\infty]$ is called a modular if for any arbitrary *x,y* in *X*

$i)\ \rho(x) = 0 \ iff \ \ x = 0$.

$ii)\ \rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$.

$iii)\ \rho(\alpha x + \beta y) \le \rho(x) + \rho(y)\ , if\ \alpha + \beta = 1\ , \alpha \ge 0, \beta \ge 0$

If we replay (i) by $\rho(\alpha x) = 0$ for all $\alpha > 0$ implies $x = 0$ then ρ called semi modular

If (iii) is replaced by $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1, \alpha \ge 0, \beta \ge 0$, then ρ is called convex modular if ρ is modular in X, then the set X_ρ given by $X_\rho = \{x \in X: \rho(\lambda X) \to 0 \text{ as } \lambda \to 0\}$ is called a modular space. X_ρ is a vector subspace of X and can be equipped with an *F-norm* defined by setting (see [12]).

$$\| x \|_\rho = \inf\{\gamma > 0 : \rho\left(\frac{x}{\gamma}\right) \le \gamma \}\ , x \in X_\rho$$

In 2008, Chistyakov [13] proposed the concept of a modular on an arbitrary set and developed the theory of metric spaces. He also introduced the idea of modular metric spaces formed by F-modular and developed the theory of these spaces produced in 2010 [14] by modular such that they were referred to be modular metric spaces.

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Definition1.2[14]: Let X be a nonempty set. A function $\mu: (0,\infty)\times X\times X\to(0,\infty)$ is said to be a metric modular on X if satisfying, for all $x,y,z\in X$ then the following condition holds:

i) $\mu_\gamma(x,y)=0$ for all $\gamma>0$ if and only if $x=y$

ii) $\mu_\gamma(x,y)=\mu_\gamma(y,x)$ for all $\gamma>0$

iii) $\mu_{\gamma+\theta}(x,y)\le\mu_\gamma(x,z)+\mu_\theta(z,y)$ for all $\gamma,\theta>0$

If instead (i), we have only condition

(i`) $\mu_\gamma(x,x)=0$ for all $\gamma>0$ then μ is said to be a (metric) pseudomodular on X.

The main property of a pseudo modular μ on a set X is following: given $x,y\in X$, the function $0<\gamma\to\mu_\gamma(x,y)\in[0,\infty]$ is nonincreasing on $(0,\infty)$.

In fact, if $0<\theta<\gamma$ then (iii), (i`) imply

$\mu_\gamma(x,y)\le\mu_{\gamma-\theta}(x,x)+\mu_\theta(x,y)=\mu_\theta(x,y)$.

In recent years, researchers have worked to develop the concept of modular, for example some fixed-point theorems for a general class of mappings in modular G-metric spaces [15], Partial modular space [16], The Meir-Keeler type contractions in extended modular b-metric spaces with an application [17] and many more [18, 19].

The idea for the study came from similar ideas in normed space such as [20, 21].

The following is a list of some common definitions used in the body of the research.

Definition1.3: Let X be a vector space, a mapping $\|.\|: X\to[0,\infty]$ is F-pesudonorm if satisfy

1) $\|x\|=0$ implies $x=0$

2) $\|\gamma x\|=\|x\|$ for all $|\gamma|=1$

3) $\|x+y\|\le\|x\|+\|y\|$ for all $x,y\in X$

4) $\|\beta_k x_k-\beta X\|\to 0$ for $\beta_k\to\beta$ and $\|x_k-x\|\to 0\ as\ k\to\infty$

Definition1.4: Let X be a vector space, a mapping $\|.\|: X\to[0,\infty]$ is s-pseudonorm or s-homogenous where $s\in(0,1]$ if satisfy

1) $\|0\|=0$

2) $\|\gamma x\|=\|x\|$ for all $|\gamma|=1$

3) $\|x+y\|\le\|x\|+\|y\|$ for all $x,y\in X$

4) $\|\gamma x\|=|\gamma|^s\|x\|$,$0<s\le 1$

2. Main Results

Definition 2.1: Let X be a vector space over the field F, a map $\rho:(0,\infty)\times X\to[0,\infty)$ is called t-pseudomodular if satisfying the following

1) $\rho_\alpha(0)=0$ for all $\alpha>0$.

2) $\rho_\alpha(\beta x)=\rho_\alpha(x)$ $for\ all\ \alpha>0\ and\ |\beta|=1$.

3) $\rho_{\alpha+\mu}(\sigma x+\beta y)\le \rho_\alpha(x)+\rho_\mu(y)$ $for\ all\ \alpha,\mu>0\ and\ \sigma,\beta\ge 0$ s.t $\sigma+\beta=1$.

If (1) replaced by $\rho_\alpha(x)=0$ for all $\alpha>0$ if and only if $x=0$ then ρ is called t-modular.

If (1) replaced by $\lim_{\alpha\to\infty}\rho(\beta x)=0$ for all $\beta>0$ then ρ is called t-semi modular.

If (3) replaced by $\rho_{\alpha+\mu}(\sigma x+\beta y)\le\sigma\rho_\alpha(x)+\beta\rho_\mu(y)$ $for\ all\ \alpha,\mu>0\ and\ \sigma,\beta\ge 0$ s.t $\sigma+\beta=1$ then ρ is called t-convex modular

Theorem 2.1: If $r>0$ then $\lim_{\alpha\to\infty}\rho_\alpha(x)=\lim_{\alpha\to\infty}\rho_{\alpha+r}(x)$

Proof: Let $\lim_{\alpha\to\infty}\rho_\alpha(x)=L$ then for all $\epsilon>0$ there exist $\mu>0$ such that $|\rho_\alpha(x)-L|<\epsilon$ for all $\alpha>\mu$ so $\alpha+r>\mu$ therefor $|\rho_{\alpha+r}(x)-$

$L| < \epsilon$ for all $\alpha > \mu$ hence $\lim_{\alpha\to\infty} \rho_{\alpha+r}(x) = L$

Proposition 2.2: If X be a vector space over the field F and ρ is t-pseudomodular on X then the set $X_\rho = \left\{ x \in X : \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0 \right\}$ is vector subspace of X

Proof: Let $x, y \in X_\rho$ to prove $x + y \in X_\rho$ i.e $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(x+y)) = 0$ .Let $\epsilon > 0$ since $x \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0$ therefor there exist δ_1 and $\mu_1 > 0$ such that if $|\gamma| < \delta_1$ then $|\rho_\alpha(\gamma x)| < \frac{\epsilon}{2}$ for all $\alpha > \mu_1$, and since $y \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma y) = 0$ therefor there exist δ_2 and $\mu_2 > 0$ such that if $|\gamma| < \delta_2$ then $|\rho_\alpha(\gamma y)| < \frac{\epsilon}{2}$ for all $\alpha > \mu_2$, . Assume $\delta = \frac{\min\{\delta_1,\delta_2\}}{2}$ so if $|\gamma| < \delta$ then $|2\gamma| < 2\delta$, take $\alpha = 2\max\{\mu_1, \mu_2\}$

$\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(x+y)) = \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_{\alpha+\alpha}(\gamma(x+y)) = \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_{\alpha+\alpha}(\frac{1}{2}2\gamma(x+y)) \leq \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(2\gamma x) + \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(2\gamma y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $\alpha > \mu$ so $\rho_\alpha(\gamma(x+y) < \epsilon$ and since $\rho_\alpha(\gamma(x+y) \geq 0$ then $|\rho_\alpha(\gamma(x+y)| < \epsilon$ for all $\alpha > \mu$ so $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(x+y)) = 0$

Therefore, $x + y \in X_\rho$.

Let $x \in X_\rho, \beta \in F$ to prove $\beta x \in X_\rho$ it is clear if $\beta = 0$ the statement holds.

If $\beta \neq 0$ let to prove $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(\beta x)) = 0$, assume $\epsilon > 0$ since $x \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0$ so there exist $\delta > 0$ and $\mu > 0$ such that if $|\gamma| < \delta$ then $|\rho_\alpha(\gamma x)| < \epsilon$ for all $\alpha > \mu$

Assume $\delta^` = \frac{\delta}{|\beta|}$ , if $|\gamma| < \delta^`$ we get $|\beta\gamma| < \delta$ then $|\rho_\alpha(\gamma\beta x)| < \epsilon$ for all $\alpha > \mu$ therefore, $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(\beta x)) = 0$ so $\beta x \in X_\rho$. Hence X_ρ is vector subspace.

Proposition2.3: Let ρ be a t-modular on X .Then:

a) $\rho_\alpha(\gamma x) = \rho_\alpha(|\gamma| x)$ for all $\alpha > 0$

b) $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) \leq \lim_{\alpha\to\infty} \rho_\alpha(x)$ for $|\gamma| = 1$

c) If $\sigma, \beta \in C$ and $|\sigma| < |\beta|$ then $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) \leq \lim_{\alpha\to\infty} \rho_\alpha(\beta x)$

d) $\rho_{\sum_{i=1}^n \alpha_i}(\sum_{i=1}^n \gamma_i x_i) \leq \sum_{i=1}^n \rho_{\alpha_i}(x_i)$ for all $\alpha_i > 0, n \geq 2$ and $\sum_{i=1}^n \gamma_i = 1$

Proof: a) Let $x \in X_\rho$ and $\gamma \in F$ then $\rho_\alpha(\gamma x) = \rho_\alpha\left(\frac{\gamma}{|\gamma|}|\gamma| x\right) = \rho_\alpha(|\gamma| x)$ (since $\left|\frac{\gamma}{|\gamma|}\right| = 1$)

b) Let $x \in X_\rho$ and $\gamma \in F$, $|\gamma| \leq 1$ then by (a) $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) = \lim_{\alpha\to\infty} \rho_{\alpha+\mu}(\gamma x) = \lim_{\alpha\to\infty} \rho_{\alpha+\alpha}(|\gamma| x) = \lim_{\alpha\to\infty} \rho_{\alpha+\alpha}(|\gamma| x + (1 - |\gamma|)0) \leq \lim_{\alpha\to\infty} \rho_\alpha(x) + \lim_{\alpha\to\infty} \rho_\alpha(0)$ = (since $\rho_\alpha(0) = 0$) therefor $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) \leq \lim_{\alpha\to\infty} \rho_\alpha(x)$

c) Let $x \in X_\rho$, $\sigma, \beta \in C$ and $|\sigma| < |\beta|$ then by (a) $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) = \lim_{\alpha\to\infty} \rho_\alpha(|\sigma| x) = \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{|\sigma|}{|\beta|}|\beta| x\right) \leq \lim_{\alpha\to\infty} \rho_\alpha(|\beta| x) = \lim_{\alpha\to\infty} \rho_\alpha(\beta x)$ therefor $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) \leq \lim_{\alpha\to\infty} \rho_\alpha(\beta x)$.

d) Take $n = 2$ then $\rho_{\sum_{i=1}^2 \alpha_i}(\sum_{i=1}^2 \gamma_i x_i) = \rho_{\alpha_1+\alpha_2}(\gamma_1 x_1 + \gamma_2 x_2) \leq \rho_{\alpha_1}(x_1) + \rho_{\alpha_2}(x_2) = \sum_{i=1}^2 \rho_{\alpha_i}(x_i)$ (since $\gamma_1 + \gamma_2 = 1$). Now we suppose the statement it is true for *n=k* and prove for *n=k+1* so $\rho_{\sum_{i=1}^{k+1} \alpha_i}(\sum_{i=1}^{k+1} \gamma_i x_i) = \rho_{\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}(\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_k x_k +$

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$\gamma_{k+1}x_{k+1}) =$
$\rho_{\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}\left(\sum_{i=1}^{k}\gamma_i\left(\frac{\gamma_1}{\sum_{i=1}^{k}\gamma_i}x_1+\cdots+\frac{\gamma_k}{\sum_{i=1}^{k}\gamma_i}x_k\right)+\gamma_{k+1}x_{k+1}\right) =$
$\rho_{\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}(\sum_{i=1}^{k}\gamma_i(\beta_1x_1+\cdots+\beta_kx_k)+\gamma_{k+1}x_{k+1}) \le \rho_{\alpha_1+\cdots+\alpha_k}(\beta_1x_1+\cdots+\beta_kx_k)+\rho_{\alpha_{k+1}}(x_{k+1}) \le \rho_{\alpha_1}(x_1)+\rho_{\alpha_2}(x_2)+\cdots+\rho_{\alpha_k}(x_k)+\rho_{\alpha_{k+1}}(x_{k+1}) = \sum_{i=1}^{k+1}\rho_{\alpha_i}(x_i)$
Hence, the statement is true for *n=k+1* and the statement true for all $n \ge 2$.

Corollary 2.4: If $0 < \alpha_1 \le \alpha_2$ then $\rho_{\alpha_1}(x) \le \rho_{\alpha_2}(x)\ for\ all\ x \in X$.

Proof: Since $\alpha_1 \le \alpha_2$ then $\alpha_2 = \alpha_1 + \beta$ $,\beta > 0$ so $\rho_{\alpha_2}(x) = \rho_{\alpha_1+\beta}(1x+0.0) \le \rho_{\alpha_1}(x)+\rho_\beta(0) = \rho_{\alpha_1}(x)$ (since $\rho_\beta(0)=0$) . Hence $\rho_{\alpha_1}(x) \le \rho_{\alpha_2}(x)\ for\ all\ x \in X_\rho$.

Corollary2.5: For every $x \in X$ the function $\rho: [0,\infty) \times X \to [0,\infty)$ is decreasing function with respect to α.

Theorem 2.6: Let $\rho: [0,\infty)\times X \to [0,\infty)$ be a t-pseudomodulr and $\| \ \|_\rho: X_\rho \to [0,\infty)$ define by $\| x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\}$ then

a) $\| x \|_\rho$ exist, for all $x \in X_\rho$

b) If $|\beta| = 1$ then $\| \beta x \|_\rho = \| x \|_\rho$ for all $x \in X_\rho$

c) $\| x+y \|_\rho \le \| x \|_\rho + \| y \|_\rho$ for all $x, y \in X_\rho$

d) If $N \ge 1$ then $\| Nx \|_\rho \le N \| x \|_\rho$ for all $x \in X_\rho$

Proof: a) To show $\| x \|_\rho$ exist we must prove the set $\{u > 0: \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\}$ non-empty Since $x \in X_\rho$ then $\rho_\alpha(\gamma x) \to 0$, as $\gamma \to 0, \alpha \to \infty$

Let $\epsilon = 1$ this implies there exist $\delta > 0$ such that if $|\gamma| < \delta$ and $\mu > 0$ then $|\rho_\alpha(\gamma x)| < 1 = \epsilon$ for all $\alpha > \mu$,because $\delta > 0$ there exist $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ hence $\rho_\alpha(\frac{1}{n}x) < 1 \le n$ therefor

$n \in \{u > 0: \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\}$, so the set $\left\{u > 0: \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\right\}$ non empty and since 0 is lower bound of the set $\{u > 0: \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\}$ then $\| x \|_\rho \ge 0$ and exist for all $x \in X_\rho$

b) $\| \beta x \|_\rho = \inf\left\{u > 0 : \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\beta x}{u}\right) \le u\right\}$
$= \inf\left\{u > 0 : \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{|\beta| x}{u}\right) \le u\right\}$ from proposition 2.3 –a
$=\inf\{u > 0 : \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right) \le u\}$
$=\| x \|_\rho$.

c) Let $x, y \in X_\rho$ then from definition of $\| x \|_\rho$ for all $\epsilon > 0$ there exist $\mu > 0$ such that $\mu < \| x \|_\rho + \epsilon$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\mu}\right) \le \mu$, since $\frac{\mu}{\| x\|_\rho+\epsilon} < 1$ then $\lim_{\alpha\to\infty}\rho_\alpha(\frac{x}{\| x\|_\rho+\epsilon}) = \lim_{\alpha\to\infty}\rho_\alpha(\frac{\mu}{\| x\|_\rho+\epsilon}\frac{x}{\mu}) \le \lim_{\alpha\to\infty}\rho_\alpha(\frac{x}{\mu}) \le \mu < \| x \|_\rho + \epsilon$, then $\lim_{\alpha\to\infty}\rho_\alpha(\frac{x}{\| x\|_\rho+\epsilon}) \le \| x \|_\rho + \epsilon$.

Similarly we get $\lim_{\alpha\to\infty}\rho_\alpha(\frac{y}{\| y\|_\rho+\epsilon}) \le \| y \|_\rho + \epsilon$. Let $u = \| x \|_\rho + \epsilon$ and $v = \| y \|_\rho + \epsilon$ then $\frac{u}{u+v}, \frac{v}{u+v} > 0$ and $\frac{u}{u+v} + \frac{v}{u+v} = 1$. To prove $u+v \in \{u > 0: \lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x+y}{u}\right) \le u\}$

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$\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x+y}{u+v}\right)=\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{u}{u+v}\frac{x}{u}+\frac{v}{u+v}\frac{y}{v}\right)\leq\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right)+\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{y}{v}\right)\leq u+v$ so $u+v\in\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x+y}{u}\right)\leq u\}$ hence $\| x+y\|_\rho\leq u+v=\| x\|_\rho+\| y\|_\rho+2\epsilon$

Since ϵ arbitrary then $\| x+y\|_\rho\leq\| x\|_\rho+\| y\|_\rho$ for all $x,y\in X_\rho$.

d) Let $\beta>0$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\beta}\right)\leq\beta$ since $N\geq 1$ then $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{1}{N\beta}Nx\right)\leq\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\beta}\right)\leq\frac{N}{N}\beta\leq N\beta$ so that $N\beta\in\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{Nx}{u}\right)\leq u\right\}$ therefor $\inf\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{Nx}{u}\right)\leq u\right\}\leq N\beta$ for all $\beta>0$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\beta}\right)\leq\beta$, so $\| Nx\|_\rho\leq N\beta$ this implies $\frac{\|Nx\|_\rho}{N}\leq\beta$ hence $\frac{\|Nx\|_\rho}{N}\leq\beta$ is lower bound for $\left\{\beta>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\beta}\right)\leq\beta\right\}$ then $\frac{\|Nx\|_\rho}{N}\leq\| x\|_\rho$ hence $\| Nx\|_\rho\leq N\| x\|_\rho$ for all $x\in X_\rho$.

Theorem 2.7: Let $\rho:[0,\infty)\times X\to[0,\infty)$ be a t-pseudomodulr and $\|\ \|_\rho:X_\rho\to[0,\infty)$ define by $\| x\|_\rho=\inf\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right)\leq u\}$ then

a) If $x_1,x_2\in X_\rho$ such that $\lim_{\alpha\to\infty}\rho_\alpha(\gamma x_1)\leq\lim_{\alpha\to\infty}\rho_\alpha(\gamma x_2)$ for all $\gamma>0$ then $\| x_1\|_\rho\leq\| x_2\|_\rho$

b) If $0\leq\gamma_1\leq\gamma_2$ then $\|\gamma_1 x\|_\rho\leq\|\gamma_2 x\|_\rho$ for all $x\in X_\rho$

c) If $\| x\|_\rho<1$ then $\lim_{\alpha\to\infty}\rho_\alpha(x)\leq\| x\|_\rho$

d) If $\| x_n\|_\rho\to$ and $\gamma\in C$ then $\|\gamma x_n\|_\rho\to 0$ $as\ n\to\infty$

Proof: a) Let $A=\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x_1}{u}\right)\leq u\right\}$, $B=\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x_2}{u}\right)\leq u\right\}$ to show $B\subseteq A$

Let $u\in B$ then $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x_2}{u}\right)\leq u$, since $\lim_{\alpha\to\infty}\rho_\alpha(\gamma x_1)\leq\lim_{\alpha\to\infty}\rho_\alpha(\gamma x_2)\leq u$ then $u\in A$ therefor $B\subseteq A$. Hence $inf A\leq inf B$ so $\| x_1\|_\rho\leq\| x_2\|_\rho$.

b) If $\gamma_1=\gamma_2$ then the statement is true

Suppose $\gamma_1<\gamma_2$ then $\frac{\gamma_1}{\gamma_2}<1$. Now let $A=\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_1 x}{u}\right)\leq u\right\}, B=\left\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_2 x}{u}\right)\leq u\right\}$ To prove $\subseteq A$, let $c\in B$ then $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_2 x}{c}\right)\leq c.$ Since $\frac{\gamma_1}{\gamma_2}<1$

$\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_1 x}{c}\right)=\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_1}{\gamma_2}\frac{\gamma_2 x}{c}\right)\leq\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{\gamma_2 x}{c}\right)\leq c$ this implies $c\in A$ so $inf A\leq inf B$

Therefor $\|\gamma_1 x\|_\rho\leq\|\gamma_2 x\|_\rho$ for all $x\in X_\rho$.

c) Since $\| x\|_\rho=\inf\{u>0:\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u}\right)\leq u\}$ and $\| x\|_\rho<1$ then there exist $u_\circ$ such that $0<u_\circ<1$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u_\circ}\right)\leq u_\circ$. Now we shall prove if $0<\gamma$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\gamma}\right)\leq\gamma$ then $\lim_{\alpha\to\infty}\rho_\alpha(x)\leq\gamma$

-If $0<\gamma\leq 1$ and $\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\gamma}\right)\leq\gamma$ then $\lim_{\alpha\to\infty}\rho_\alpha(x)=\lim_{\alpha\to\infty}\rho_\alpha\left(\gamma\frac{x}{\gamma}\right)\leq\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{\gamma}\right)\leq\gamma$

-If $\gamma>1$ then $\lim_{\alpha\to\infty}\rho_\alpha(x)=\lim_{\alpha\to\infty}\rho_\alpha\left(u_\circ\frac{x}{u_\circ}\right)\leq\lim_{\alpha\to\infty}\rho_\alpha\left(\frac{x}{u_\circ}\right)\leq u_\circ\leq 1<\gamma$.Therefor $\lim_{\alpha\to\infty}\rho_\alpha(x)$ is lower bound for the set

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$\{u > 0: \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \leq u\}$, hence $\lim_{\alpha\to\infty} \rho_\alpha \leq \inf\{u > 0 : \rho_\alpha\left(\frac{x}{u}\right) \leq u , \forall\, \alpha \geq 0\}$

Therefor $\lim_{\alpha\to\infty} \rho_\alpha \leq \| x \|_\rho$.

d)Let N positive integer number and $|\gamma| < N$ since $\gamma \leq |\gamma| < N$ then $\gamma \leq N$ so by (b)

$\| \gamma x_n \|_\rho \leq \| N x_n \|_\rho \leq N \| x_n \|_\rho \to 0 \;\; as\; n \to \infty$

Therefor $\| \gamma x_n \|_\rho \to 0 \;\; as\; n \to \infty$

Proposition 2.8: If $\rho: [0,\infty) \times X \to [0,\infty)$ is t-semi modular the function which define by

$\| x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \leq u \}$ is F-norm on X_ρ.

Proof: 1) If $x = 0$ then $\| 0 \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{0}{u}\right) \leq u \} = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (0) \leq u \}$

$= \inf\{u > 0 : 0 \leq u \} = \inf (0,\infty) = 0$.

Conversely let $\| x \|_\rho = 0$ then $\| x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \leq u \} = 0$ then there exist

$u_n \geq 0$ such that $u_n \to 0\; as\; n \to \infty$ and $\lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u_n}\right) \leq u_n \;\; \forall n$

Let γ positive integer, since $u_n \to 0$ then there exist positive integer k such that $\gamma u_n < 1 \;\; \forall n > k$

Since ρ is t-semi modular then $\lim_{\alpha\to\infty} \rho_\alpha (\gamma x) = \lim_{\alpha\to\infty} \rho_\alpha\left(\gamma u_n \frac{x}{u_n}\right) \leq \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{\gamma}{u_n}\right) \leq u_n \to 0$ for all $n > k$ So $\lim_{\alpha\to\infty} \rho_\alpha (\gamma x) = 0 \;\; \forall \gamma > 0$, hence $x = 0$

2) $\| \gamma\, x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{\gamma x}{u}\right) \leq u \}$

$= \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{|\gamma| x}{u}\right) \leq u \}$

from proposition 2.3-a

$= \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \leq u \}$

$= \| x \|_\rho$

3) similarly proposition 2.6-c

4) Let $\beta_n \to \beta$ and $\|x_n - x \|_\rho \to 0 \;\; as\; n \to \infty$ to prove $\|\beta_n x_n - \beta x\| \to 0\; as\; n \to \infty$

Writing $c_n = \beta_n - \beta$ and $y_n = x_n - x$ we have $c_n \to 0$ and $y_n \to 0\; as\; n \to \infty$

Now give $\epsilon > 0$ and since $x \in X_\rho$, we get $\rho_\alpha\left(c_n \frac{x}{\epsilon}\right) \to 0 \;\; as\; n \to \infty$ so $\rho_{\alpha_x}\left(c_n \frac{x}{\epsilon}\right) < \epsilon$

Hence $\|c_n x \| \to 0$, taking N positive integer such that $|\beta_n| \leq N \;\; \forall\, n$ we have

$\|\beta_n x_n - \beta x\|_\rho = \|\beta_n x_n - \beta_n x + \beta_n x - \beta x\|_\rho \leq \|\beta_n x_n - \beta_n x\|_\rho + \| \beta_n x - \beta x\|_\rho = \|\beta_n (x_n - x)\|_\rho + \| (\beta_n - \beta) x\| \leq \|N y_n \| + \|c_n x\| \leq N\|y_n \| + \|c_n x\| \to 0 \;\; as\; n \to \infty$

Hence $\|\beta_n x_n - \beta x\|_\rho \to 0\; as\; n \to \infty$

Therefor $\| x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \leq u \}$ is F-norm on X_ρ.

Definition 2.2: Let X be a vector space over the field F, a map $\rho: [0,\infty) \times X \to [0,\infty)$ is called t-s-convex modular if satisfy the following

1) $\rho_\alpha(x) = 0$ iff $x = 0$.

2) $\rho_\alpha(\beta x) = \rho_\alpha(x) \;\; for\ all\ \alpha > 0\ and\ |\beta| = 1$.

3) $\rho_{\alpha+\mu}(\sigma x + \beta y) \leq \sigma \rho_\alpha(x) + \beta \rho_\mu(y) \;\; for\ all\ \alpha, \mu > 0\ and\ \sigma, \beta \geq 0$ s.t $\sigma + \beta = 1$ for $s \in (0,1]$.

Theorem 2.9: If X be a vector space over the field F and ρ is t-s-convex modular on X then the set $X_\rho = \left\{ x \in X : \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0 = 0 \right\}$ is vector subspace of X

Proof: Let $x, y \in X_\rho$ to prove $x + y \in X_\rho$,

Let $\epsilon > 0$ since $x \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0$ therefor there exist δ_1 and μ_1 such that if $|\gamma| < \delta_1$ then $|\rho_\alpha(\gamma x)| < (\frac{1}{2})^{-s+1}\epsilon$ for all $\alpha > \mu_1$, and since $y \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma y) = 0$ therefor there exist δ_2 and $\mu_2 > 0$such that if $|\gamma| < \delta_2$ then $|\rho_\alpha(\gamma y)| < (\frac{1}{2})^{-s+1}\epsilon$ for all $\alpha > \mu_2$. Assume $\delta = \frac{\min\{\delta_1,\delta_2\}}{2}$ so if $|\gamma| < \delta$ then $|2\gamma| < 2\delta$ and let $\mu = 2\max\{\mu_1, \mu_2\}$

$\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(x + y)) = \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_{\alpha+\alpha}\left(\frac{1}{2}2\gamma x + \frac{1}{2}2\gamma y\right) \le$

$(\frac{1}{2})^s \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(2\gamma x) +$

$(\frac{1}{2})^s \lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(2\gamma y) < (\frac{1}{2})^s(\frac{1}{2})^{-s+1}\epsilon +$

$(\frac{1}{2})^s(\frac{1}{2})^{-s+1}\epsilon = \epsilon$ so $\rho_\alpha(\gamma(x + y) < \epsilon$ for all $\alpha > \mu$ and since $\rho_\alpha(\gamma(x + y) \ge 0$ then $|\rho_\alpha(\gamma(x + y)| < \epsilon$ so $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(x + y)) = 0$ therefore $x + y \in X_\rho$.

Let $x \in X_\rho$, $\beta \in F$ to prove $\beta x \in X_\rho$ it is clear if $\beta = 0$ the statement holds .

If $\beta \neq 0$ to prove $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma(\beta x))$, assume $\epsilon > 0$ since $x \in X_\rho$ then $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0$ so there exist $\delta > 0$ and $\mu > 0$ such that if $|\gamma| < \delta$ then $|\rho_\alpha(\gamma x)| < \epsilon$ for all $\alpha > \mu$. Take $\delta^` = \frac{\delta}{|\beta|}$ , if $|\gamma| < \delta^`$ we get $|\beta\gamma| < \delta$ then $|\rho_\alpha(\gamma\beta x)| < \epsilon$ for all $\alpha > \mu$ therefor $\lim_{\substack{\gamma\to 0\\ \alpha\to\infty}} \rho_\alpha(\gamma x) = 0$. So $\beta x \in X_\rho$

Hence X_ρ is vector subspace.

Proposition2.10: Let ρ be a t-s-convex modular on X and $0 < s \le 1$ then

a) $\rho_\alpha(\gamma x) = \rho_\alpha(|\gamma| x)$ $for\ all\ \alpha > 0$

b) $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) \le |\gamma|^s \lim_{\alpha\to\infty} \rho_\alpha(x)$ for $|\gamma| \le 1$

c) If $\sigma, \beta \in C\ and\ |\sigma| < |\beta|$ then $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) \le \lim_{\alpha\to\infty} \rho_\alpha(\beta x)$

d) $\rho_{\sum_{i=1}^n \alpha_i}(\sum_{i=1}^n \gamma_i x_i) \le \sum_{i-1}^n (\gamma_i)^s \rho_{\alpha_i}(x_i)$ for all $\alpha_i \ge 0$, $n \ge 2$ and $\sum_{i=1}^n (\gamma_i)^s = 1$

Proof: a) similarly proof of proposition 2.3-a

b) Let $x \in X_\rho\ and\ \gamma \in F$, $|\gamma| \le 1$ then by (a) $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) = \lim_{\alpha\to\infty} \rho_\alpha(|\gamma| x) =$
$\lim_{\alpha\to\infty} \rho_\alpha(|\gamma| x + (1 - |\gamma|)0) \le$
$|\gamma|^s \lim_{\alpha\to\infty} \rho_\alpha(x) +$
$(1 - |\gamma|)^s \lim_{\alpha\to\infty} \rho_\alpha(0) = |\gamma|^s \lim_{\alpha\to\infty} \rho_\alpha(x)$
(since $\lim_{\alpha\to\infty} \rho_\alpha(0) = 0$)

Therefor $\lim_{\alpha\to\infty} \rho_\alpha(\gamma x) \le |\gamma|^s \lim_{\alpha\to\infty} \rho_\alpha(x)$.

c) Let $x \in X_\alpha$, $\sigma, \beta \in C\ and\ |\sigma| < |\beta|$ then by (a) $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) = \lim_{\alpha\to\infty} \rho_\alpha(|\sigma| x) =$
$\lim_{\alpha\to\infty} \rho_\alpha\left(\frac{|\sigma|}{|\beta|}|\beta| x\right) \le$

$(\frac{|\sigma|}{|\beta|})^s \lim_{\alpha\to\infty} \rho_\alpha(|\beta| x) \le \lim_{\alpha\to\infty} \rho_\alpha(|\beta| x) =$
$\lim_{\alpha\to\infty} \rho_\alpha(\beta x)$ therefor $\lim_{\alpha\to\infty} \rho_\alpha(\sigma x) \le \lim_{\alpha\to\infty} \rho_\alpha(\beta x)$.

d) Take $n = 2$ then $\rho_{\sum_{i=1}^2 \alpha_i}\left(\sum_{i=1}^2 \gamma_i x_i\right) = \rho_{\alpha_1+\alpha_2}(\gamma_1 x_1 + \gamma_2 x_2) \le {\gamma_i}^s \rho_{\alpha_1}(x_1) + {\gamma_i}^s \rho_{\alpha_2}(x_2) = \sum_{i=1}^2 {\gamma_i}^s \rho_{\alpha_i}(x_i)$ (since ${\gamma_1}^s + {\gamma_2}^s = 1$). Now we suppose the statement it is true for *n=k* and prove for *n=k+1* so

$\rho_{\sum_{i=1}^{k+1} \alpha_i}\left(\sum_{i=1}^{k+1} \gamma_i x_i\right) =$
$\rho_{\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}(\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_k x_k + \gamma_{k+1} x_{k+1}) =$

$\rho_{\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}\left(\sum_{i=1}^k ({\gamma_i}^s)^{\frac{1}{s}}\left((\frac{{\gamma_1}^s}{\sum_{i=1}^k {\gamma_i}^s})^{\frac{1}{s}} x_1 +\right.\right.$

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$\dots + (\frac{\gamma_k{}^s}{\sum_{i=1}^k \gamma_i{}^s})^{\frac{1}{s}} x_k\Big) + \gamma_{k+1} x_{k+1}\Big) =$

$\rho_{\alpha_1+\alpha_2+\dots+\alpha_k+\alpha_{k+1}} \Big(\sum_{i=1}^k (\gamma_i{}^s)^{\frac{1}{s}} (\beta_1 x_1 + \dots +$

$\beta_k x_k) + \gamma_{k+1} x_{k+1}\Big) \leq$

$\sum_{i=1}^k (\gamma_i)^s \rho_{\alpha_1+\dots+\alpha_k}(\beta_1 x_1 + \dots + \beta_k x_k) +$

$\gamma_{k+1}{}^s \rho_{\alpha_{k+1}}(x_{k+1}) \leq \sum_{i=1}^k (\gamma_i)^s (\beta_1{}^s \rho_{\alpha_1}(x_1) +$

$\beta_2{}^s \rho_{\alpha_2}(x_2) + \dots + \beta_k{}^s \rho_{\alpha_k}(x_k) +$

$\gamma_{k+1}{}^s \rho_{\alpha_{k+1}}(x_{k+1}) =$

$\sum_{i=1}^k (\gamma_i)^s \left(\frac{\gamma_1{}^s}{\sum_{i=1}^k \gamma_i{}^s} \rho_{\alpha_1}(x_1) + \dots + \right.$

$\left.\frac{\gamma_k{}^s}{\sum_{i=1}^k \gamma_i{}^s} \rho_{\alpha_k}(x_k)\right) + \gamma_{k+1}{}^s \rho_{\alpha_{k+1}}(x_{k+1}) =$

$\gamma_1{}^s \rho_{\alpha_1}(x_1) + \dots + \gamma_k{}^s \rho_{\alpha_k}(x_k) +$

$\gamma_{k+1}{}^s \rho_{\alpha_{k+1}}(x_{k+1}) = \sum_{i=1}^{k+1} \gamma_i{}^s \rho_{\alpha_i}(x_i)$ hence the statement it is true for *n=k+1* then the statement true foe all $n \geq 2$

Theorem 2.11: Let ρ be t-s-convex modular, and $\|.\|_\rho^s : X_\rho \to [0,\infty)$ define by

$\|x\|_\rho^s = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \leq 1\}$ then the following holds

a) If $\sigma, \beta \in C$ such that $|\sigma| \leq |\beta|$ then $\|\sigma x\|_\rho^s \leq \|\beta x\|_\rho^s$

b) $\|x + y\|_\rho^s \leq \|x\|_\rho^s + \|y\|_\rho^s$ for all $x, y \in X_\rho$

c) If N positive integer then $\|Nx\|_\rho^s \leq N\|x\|_\rho^s$

Proof: a) If $\sigma = \beta$ then the statement holds

If $\sigma < \beta$ then $\frac{\sigma}{\beta} < 1$ also $(\frac{\sigma}{\beta})^s < 1$ for $\in (0,1]$.

Now to prove $\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{u^{\frac{1}{s}}}) \leq 1\} \subset \{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\sigma x}{u^{\frac{1}{s}}}) \leq 1\}$, let $\gamma \in \{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{u^{\frac{1}{s}}}) \leq 1\}$ so $\lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{\gamma^{\frac{1}{s}}}) \leq 1$. Then

$\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{\sigma x}{\gamma^{\frac{1}{s}}}\right) \leq \lim_{\alpha\to\infty} \rho_\alpha (\frac{\sigma}{\beta} \frac{\beta x}{\gamma^{\frac{1}{s}}}) \leq$

$(\frac{\sigma}{\beta})^s \lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{\gamma^{\frac{1}{s}}}) \leq \lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{\gamma^{\frac{1}{s}}}) \leq 1$

(since $(\frac{\sigma}{\beta})^s < 1$)

Hence $\gamma \in \{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\sigma x}{u^{\frac{1}{s}}}) \leq 1\}$ i.e $\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\sigma x}{u^{\frac{1}{s}}}) \leq 1\} \leq inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{\beta x}{u^{\frac{1}{s}}}) \leq 1\}$. Therefor $\|\sigma x\|_\rho^s \leq \|\beta x\|_\rho^s$

b) Let $\epsilon > 0$ and since $\|x\|_\rho^s = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \leq 1\}$ then there exist $\mu > 0$ such that $\mu < \|x\|_\rho^s + \epsilon$ and $\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{\mu}\right) \leq \mu$, since $\frac{\mu}{\|x\|_\rho^s + \epsilon} < 1$ then $\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{\|x\|_\rho^s + \epsilon}\right) =$

$\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{\mu}{\|x\|_\rho^s + \epsilon} . \frac{x}{\mu}\right) \leq \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{\mu}\right) \leq \mu <$

$\|x\|_\rho^s + \epsilon$, then $\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{\|x\|_\rho^s + \epsilon}\right) \leq \|x\|_\rho^s + \epsilon$.

Similarly we get $\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{y}{\|y\|_\rho^s + \epsilon}\right) \leq \| y \|_\rho + \epsilon$. Let $u = \|x\|_\rho^s + \epsilon$ and $v = \|y\|_\rho^s + \epsilon$ then $\frac{u}{u+v}, \frac{v}{u+v} > 0$ and $\frac{u}{u+v} + \frac{v}{u+v} = 1$. To prove $u + v \in \{u > 0 : \rho_{\alpha_{x+y}} (\frac{x+y}{u}) \leq u\}$ take $\alpha_{x+y} = \alpha_x + \alpha_y$

$\lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x+y}{(u+v)^{\frac{1}{s}}}\right) =$

$\lim_{\alpha\to\infty} \rho_{\alpha+\alpha} \left(\frac{u^{\frac{1}{s}}}{(u+v)^{\frac{1}{s}}} \frac{x}{u^{\frac{1}{s}}} + \frac{v^{\frac{1}{s}}}{(u+v)^{\frac{1}{s}}} \frac{y}{v^{\frac{1}{s}}}\right) \leq$

$\frac{u}{u+v} \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{u}\right) + \frac{v}{u+v} \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{y}{v}\right) \leq$

$\frac{u}{u+v} + \frac{v}{u+v} \leq 1$ so

$u + v \in \{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x+y}{u}) \leq u\}$ hence $\|x + y\|_\rho^s \leq u + v = \|x\|_\rho^s + \|y\|_\rho^s + 2\epsilon$, hence

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$\|x+y\|_\rho^s \le \|x\|_\rho^s + \|y\|_\rho^s$.

c) $\|Nx\|_\rho^s = \|x+x+\dots+x\|_\rho^s$ N-times

$\le N\|x\|_\rho^s$

Thererom2.12: Let ρ be t-s-convex modular, and $\|.\|_\rho^s : X_\rho \to [0,\infty)$ define by $\|x\|_\rho^s = \inf\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \le 1\}$ then the following holds

a) If $x_1, x_2 \in X_\rho$ such that $\lim_{\alpha\to\infty} \rho_\alpha (\gamma x_1) \le \lim_{\alpha\to\infty} \rho_\alpha (\gamma x_2)$ for all $\gamma > 0$ then $\|x_1\|_\rho^s \le \|x_2\|_\rho^s$

b) If $\|x\|_\rho^s < 1$ then $\lim_{\alpha\to\infty} \rho_\alpha \le \|x\|_\rho^s$

c) If $\|x_n\|_\rho^s \to 0$ and $\gamma \in C$ then $\|\gamma x_n\|_\rho^s \to 0$ as $n \to \infty$

Proof: a) Let $= \{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x_1}{u^{\frac{1}{s}}}) \le 1\}$, $B = \{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x_2}{u^{\frac{1}{s}}}) \le 1\}$ to show $B \subseteq A$

Let $v \in B$ so $v > 0$ and $\lim_{\alpha\to\infty} \rho_\alpha (\frac{x_2}{v^{\frac{1}{s}}}) \le 1$

since $\lim_{\alpha\to\infty} \rho_\alpha (\frac{x_1}{u^{\frac{1}{s}}}) \le \lim_{\alpha\to\infty} \rho_\alpha (\frac{x_2}{v^{\frac{1}{s}}}) \le 1$ then $v \in A$ hence $B \subseteq A$

Therefor $\inf A \le \inf B$ i.e $\|x_1\|_\rho^s \le \|x_2\|_\rho^s$

b) Since $\inf\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \le 1\} < 1$ then there exist $0 < v < 1$ such that $\lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{v^{\frac{1}{s}}}) \le 1$ so

$\lim_{\alpha\to\infty} \rho_\alpha (x) = \lim_{\alpha\to\infty} \rho_\alpha (v^{\frac{1}{s}} \frac{x}{v^{\frac{1}{s}}}) \le$

$v \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{v^{\frac{1}{s}}}) \le v < 1$ then $\lim_{\alpha\to\infty} \rho_\alpha (x) < 1$ .Now let $\gamma \in \{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \le 1\}$

If $0 < \gamma \le 1$ and $\lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{\gamma^{\frac{1}{s}}}) \le 1$ so

$\lim_{\alpha\to\infty} \rho_\alpha (x) = \lim_{\alpha\to\infty} \rho_\alpha (\gamma^{\frac{1}{s}} \frac{x}{\gamma^{\frac{1}{s}}}) \le$

$\gamma \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{\gamma^{\frac{1}{s}}}) \le \gamma$ ……(1)

If $\gamma > 1$ since $\rho_{\alpha_x}(x) < 1$ we get $\lim_{\alpha\to\infty} \rho_\alpha (x) < 1 < \gamma$ …(2)

From (1) ,(2) we get $\lim_{\alpha\to\infty} \rho_\alpha (x)$ is lower bound for the set $\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \le 1\}$

hence $\lim_{\alpha\to\infty} \rho_\alpha (x) \le \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{u^{\frac{1}{s}}}\right) \le 1\right\} = \|x\|_\rho^s$ therefor $\lim_{\alpha\to\infty} \rho_\alpha (x) \le \|x\|_\rho^s$

c)Similarly proof of proposition 2.7-d

Theorem 2.12: If ρ is t-s-convex pseudomodular and, $0 < s \le 1$ then $\|x\|_\rho^s = \inf\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (\frac{x}{u^{\frac{1}{s}}}) \le 1\}$ is s-homogenous

Proof: 1)

$\|0\|_\rho^s = \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{0}{u^{\frac{1}{s}}}\right) \le 1\right\} =$

$\inf\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha (0) \le 1\} =$

$\inf\{u>0 : 0 \le 1\} = \inf(0,\infty) = 0$

2) $\|\gamma x\|_\rho^s = \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{\gamma x}{u^{\frac{1}{s}}}\right) \le 1\right\} = \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{|\gamma| x}{u^{\frac{1}{s}}}\right) \le 1\right\} =$

$\inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{x}{u^{\frac{1}{s}}}\right) \le 1\right\} = \|x\|_\rho^s$

3) similarly proof of theorem 2.11-b

4) $\|\gamma x\|_\rho^s = \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{\gamma x}{u^{\frac{1}{s}}}\right) \le 1\right\} = \inf\left\{u>0 : \lim_{\alpha\to\infty} \rho_\alpha \left(\frac{|\gamma| x}{u^{\frac{1}{s}}}\right) \le 1\right\} =$

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$|\gamma|^s \inf\left\{\frac{u}{|\gamma|^s} > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{\left(\frac{\alpha}{|\gamma|^s}\right)^{\frac{1}{s}}}\right) \le 1\right\} =$

$|\gamma|^s \|x\|$

Therefor $\|x\|_\rho^s = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u^{\frac{1}{s}}}\right) \le 1\}$ is s-homogenous

Theorem2.14: If ρ is t-s-convex modular and, $0 < s \le 1$ then $\|x\|_\rho^s = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u^{\frac{1}{s}}}\right) \le 1\}$ is s-norm.

3. Conclusion

The main result that we obtained from this article is the definition of t-modular and the good results that resulted from it, including the definition of the norm in the form $\| x \|_\rho = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u}\right) \le u\}$, and it was proven that this norm forms F-norm when ρ is t-semi modular.

In addition to that, we defined t-s-convex modular, and through it we proved that the norm defined as $\|x\|_\rho^s = \inf\{u > 0 : \lim_{\alpha\to\infty} \rho_\alpha\left(\frac{x}{u^{\frac{1}{s}}}\right) \le 1\}$ forms s-homogenous.

One of the results worth noting is that the t-modular function is a decreasing function.

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6. Author contribution: Authors contributed equally in the study.

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