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Orbitally Stability of Log-Logistic Autoregressive Model with Application

Mohammed A. Hamad, Azher A. Mohammad
Mathematics Department, College of Computer Science and Mathematics, Tikrit University, Iraq

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ABSTRACT

This research aims to study and finding the conditions for stability of the limit cycle of the proposed model (Log-Logistic autoregressive) based on the cumulative function of the Log-Logistic distribution. We first proved the conditions for the first order orbital stability with period ($q > 1$), and then generalized the conditions for orbital stability of order p to Log-Logistic AR (p). We give some examples of the proven conditions, and then we plot the trajectories by using different initial values.

Keywords: Stability, Log-Logistic autoregressive model, Limit cycle, Non-linear time series, Local linearization technique.

Name: **Mohammed A. Hamad** **E-mail:** mohammed.a.hamadmm2318@st.tu.edu.iq

الاستقرار المداري لأنموذج الانحدار الذاتي اللوغاريتمي اللوجستي

محمد علي حمد، أزهر عباس محمد
قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة تكريت، تكريت، العراق

الملخص

يهدف هذا البحث إلى دراسة وإيجاد شروط استقرارية دورة النهاية لأنموذج لوغاريتم اللوجيستي للانحدار الذاتي والمبني على اساس الدالة التراكمية لتوزيع (Log-Logistic). لقد أثبتنا أولاً شروط الاستقرار المداري من الرتبة الأولى مع الدورة ($q > 1$) , ثم قمنا بتعميم شروط الاستقرار المداري على نفس الأنموذج من الرتبة p(Log-Logistic AR(P)). ثم تم إعطاء بعض الأمثلة على الشروط المثبتة، وقمنا برسم مسارات الحل باستخدام قيم ابتدائية مختلفة.

1. INTRODUCTION

In this study we suggest one of a non-linear autoregressive model, the Log-Logistic Autoregressive model with order p, often known as the Log-Logistic AR(p) model. It is dependent upon the Log-Logistic distribution's cumulative distribution function. We will use the local linearization approximation to analyze and finding the stability conditions for a limit cycle in this non-linear autoregressive model of order one when period (q > 1), and then generalized the conditions for orbital stability of order p to Log-Logistic AR (p). We will also provide some examples of Log-Logistic AR (1) and demonstrate when the model is orbitally stable or not. And by use the state space representation to analyze the stability conditions for a limit cycle with period q. Some opportunities have been provided to study the stability conditions of the limit cycle of several nonlinear time series models. In 1977 has been the exponential autoregressive model was explored and studied the stability conditions of this model (1). Also has been studied the statistical analysis of turbulent limit cycle processes using nonlinear time series models in 1982 (2). In 2007 has been conducted study on stability of logistic autoregressive model and find stability of limit cycle of this model. In (2018) has been investigated the non-Linear exponential autoregressive model's stability (3). In (2019) has been conducted research on a stability of a nonlinear model`s with a hyperbolic secant function (4). In (2020) has been studied the stability by using local linearization method (5). And in the same year has been studied dynamical approach in studying stability condition of exponential (GARCH) models (6). In (2021) has been studied dynamical approach in studying GJR-GARCH (Q, P) models with application (7). As studied in (2022) the stability of the double exponential autoregressive model with the application (8). In (2023) has been studied the stability of SATER models in a dynamic approach with application (9). And in same year have been studied stability conditions of limit cycle for Gompertz autoregressive model and also stability conditions for a nonlinear time series model (9, 10). In (2024) has been a new analytical study of prey–predator dynamical systems involving the effects of Hide-and-Escape and predation skill augmentation (11). In this research, we investigate and determine the stability conditions of a limit cycle for a nonlinear Log-Logistic autoregressive model. Also provide several examples to elucidate the orbital stable and orbital unstable states using plots of trajectories with different initial values.

2. PRELIMINARIES

The Log-Logistic distribution has been studied by Shah and Dave in (1963) (12). This distribution is continuous probability for a non-negative random variable is called (the "Fisk distribution" in economics). It resembles the Log-normal distribution but has thicker tails. $(\alpha > 0)$ is a scale parameter that also represents the distribution's median. The parameter $(\beta > 0)$ represents the shape parameter.

The function of the cumulative distribution is provided by:

$$F(x;\alpha,\beta) = \frac{1}{1+(\alpha/x)^{\beta}}$$

where $x > 0\,,\alpha > 0\,,\beta > 0$

The graph of the (c.d.f) Log-Logistic distribution with different values of β is depicted in Figure (1)(13, 14).

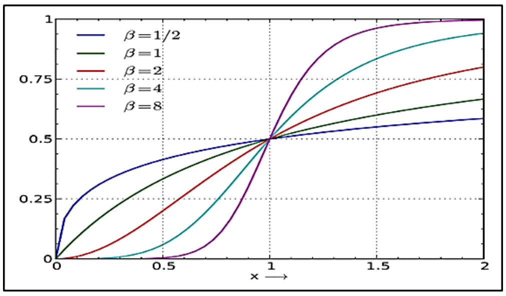

Fig 1. Graph the [c.d.f.] of Log-Logistic distribution with α=1 and different value of β.

2.1. Definition

The Log-Logistic AR (p) model given by a discrete time series $\{x_t\}$ defined as

$$x_t = \sum_{i=1}^{p}\left[\varphi_i + \frac{\pi_i}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}\right] x_{t-i} + z_t, \quad \dots (1)$$

where $\{z_t\}$ represents a white noise process, z_t has identically independent normal distribution with mean zero and variance σ_z^2, $z_t \sim iidN(0, \sigma_z^2)$, α denote a scale parameter and β denote a shape parameter, and $\{\varphi_i\}$, $\{\pi_i\}$ are constants since $i = 1,2,\dots,p$

Now we study the conditions of stability of the limit cycle for the Log-Logistic AR (1) model with a period $q > 1$ is found in the following proposition:

2.1. Proposition 1

When the following Log-Logistic AR (1) model possess a limit cycle of period $q > 1$

$$x_t = \left[\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}\right] x_{t-1} + z_t \quad \dots (2)$$

Then the model of equation (2.2) is orbital stable if

$$\left|\prod_{i=1}^{q}\left[\varphi_1 + \pi_1\left[\frac{1+(1+\beta)\left(\frac{x_{t+q-i}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t+q-i}}{\alpha}\right)^{-\beta}\right)^2}\right]\right| < 1 \quad \dots (3)$$

2.1.1. Proof

First, we find the non-zero singular point of model by putting

$x_t = x_{t-1} = x_{t-2} = \cdots = x_{t-q} = \xi$ after nigligate the effect of a white noise process $\{z_t\}$ where we assume that $z_t = 0$. The model Log-Logistic AR (1) because

$$\xi = \sum_{i=1}^{p}\left[\varphi_i + \frac{\pi_i}{1+\left(\frac{\xi}{\alpha}\right)^{-\beta}}\right]\xi$$

$$1 = \sum_{i=1}^{p}\varphi_i + \sum_{i=1}^{p}\frac{\pi_i}{1+\left(\frac{\xi}{\alpha}\right)^{-\beta}}$$

$$1 - \sum_{i=1}^{p}\varphi_i = \sum_{i=1}^{p}\frac{\pi_i}{1+\left(\frac{\xi}{\alpha}\right)^{-\beta}}$$

$$1 + \left(\frac{\xi}{\alpha}\right)^{-\beta} = \frac{\sum_{i=1}^{p}\pi_i}{1-\sum_{i=1}^{p}\varphi_i}$$

Let $k = \frac{\sum_{i=1}^{p}\pi_i}{1-\sum_{i=1}^{p}\varphi_i}$ $\quad \dots (4)$

$$\left(\frac{\xi}{\alpha}\right)^{-\beta} = k-1$$

$$\frac{\xi}{\alpha} = (k-1)^{-\frac{1}{\beta}}$$

$$\xi = \alpha(k-1)^{-\frac{1}{\beta}} \quad \text{therefore } \xi = \frac{\alpha}{\sqrt[\beta]{(k-1)}} \quad \dots (5)$$

Of them non-zero singular point exists if and only if

$$k-1 > 0 \text{ then } k > 1 \text{ or } \frac{\sum_{i=1}^{p}\pi_i}{1-\sum_{i=1}^{p}\varphi_i} > 1 \quad \dots (6)$$

Suppose that the Log-Logistic AR (1) possess a limit cycle of period $q > 1$ that is the limit cycle can be written as

$$x_t, x_{t+1}, \dots, x_{t+q} = x_t \quad \dots (7)$$

near each point of a limit cycle with sufficiently small radius ξ_s of its neighbourhood we use an equation

$x_s = x_s + \xi_s$ for $s = t, t-1, t-2, \dots, t-q$

such that $\xi_s^n \to 0$ for $n \geq 2$

by substituting x_s in equation (1) by $x_s + \xi_s$ and we get after neglect a white noise process where we assume that $z_t = 0$.

$$x_t + \xi_t = \left[\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}+\xi_{t-1}}{\alpha}\right)^{-\beta}}\right](x_{t-1} + \xi_{t-1})$$

$\dots (8)$

simplify the term $\frac{1}{1+\left(\frac{x_{t-1}+\xi_{t-1}}{\alpha}\right)^{-\beta}}$ as follows

$$\frac{1}{1+\left(\frac{x_{t-1}+\xi_{t-1}}{\alpha}\right)^{-\beta}} = \frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(1+\frac{\xi_{t-1}}{x_{t-1}}\right)^{-\beta}}$$

and by using a Taylor expansion $(1+x)^{-k} = \sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(k+j)}{j!\,\Gamma(k)}x^j$, $k > 0$

We get

$$\frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(1+\frac{\xi_{t-1}}{x_{t-1}}\right)^{-\beta}} =$$

$$\frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(1-\beta\frac{\xi_{t-1}}{x_{t-1}}+\frac{\beta(\beta-1)}{2!}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)^2-\dots\right)}$$

Since $\xi_{t-1}^n \to 0$ for $n \geq 2$

$$= \frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(1-\beta\frac{\xi_{t-1}}{x_{t-1}}\right)} = \frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}-\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}$$

$$= \frac{1}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)-\left(\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)\right)} * \frac{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)+\left(\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)+\left(\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)\right)}$$

$$= \frac{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}+\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2-\left(\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)\right)^2}$$ since

$\xi_{t-1}^n \to 0$ for $n \geq 2$

$$= \frac{1+(\frac{x_{t-1}}{\alpha})^{-\beta}+\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}} = \frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}} \quad \dots (9)$$

We substitute (9) in (8) and we well get

$$x_t + \xi_t = [\varphi_1 + \pi_1[\frac{1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}]](x_{t-1} + \xi_{t-1})$$

$$x_t + \xi_t = [\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}](x_{t-1} + \xi_{t-1})$$

$$x_t + \xi_t = [\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}]\, x_{t-1} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}} x_{t-1} + [\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\frac{\xi_{t-1}}{x_{t-1}})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}]\, \xi_{t-1}$$

Since $\quad x_t = [\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}]\, x_{t-1}$

$$x_t + \xi_t = x_t + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}} x_{t-1} + [\,\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}]\, \xi_{t-1}$$

$$\xi_t = \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}(\xi_{t-1})}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}} + [\,\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}]\xi_{t-1}$$

$$\xi_t = \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}\xi_{t-1} + [\,\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}]\xi_{t-1} + [\frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\left(\frac{\xi_{t-1}}{x_{t-1}}\right)}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}]\xi_{t-1}$$

since $\xi_{t-1}^{n} \to 0$ for $n \geq 2$

$$\xi_t = [\,\varphi_1 + \frac{\pi_1}{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}} + \frac{\pi_1\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}\,]\xi_{t-1}$$

$$\xi_t = [\,\varphi_1 + \pi_1[\frac{1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}+\beta\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}\,]\,]\xi_{t-1}$$

$$\xi_t = [\,\varphi_1 + \pi_1[\frac{1+(1+\beta)\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^{2}}\,]\,]\xi_{t-1} \quad \dots (10)$$

This equation is difference equation with non-constant coefficient at its difficult to solve it analytically but we can see the convergent condition by showing when the ratio $\left|\frac{\xi_{t+q}}{\xi_t}\right|$ approach to zero or not, that is $\left|\frac{\xi_{t+q}}{\xi_t}\right| < 1$

from (10) let $T(x_s) = \varphi_1 + \pi_1[\frac{1+(1+\beta)\left(\frac{x_s}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_s}{\alpha}\right)^{-\beta}\right)^{2}}\,]$,

$s = t, t-1, t-2, \dots, t-q$

that is $\xi_t = T(x_{t-1})\,\xi_{t-1}$

and so on

$\xi_{t+1} = T(x_t)\,\xi_t = T(x_t)\,T(x_{t-1})\,\xi_{t-1} =$
$T(x_t)T(x_{t-1})\,T(x_{t-2})\,\xi_{t-2}\ \dots$

for $q > 1$ we get

$\xi_{t+q} = T\left(x_{t+q-1}\right)\xi_{t+q-1} =$
$T\left(x_{t+q-1}\right)T\left(x_{t+q-2}\right)\xi_{t+q-2} = \cdots$

then $\quad \xi_{t+q} = \prod_{i=1}^{q} T\left(x_{t+q-i}\right)\xi_t$

$$\left|\frac{\xi_{t+q}}{\xi_t}\right| = \left|\prod_{i=1}^{q} T\left(x_{t+q-i}\right)\right| \quad \dots (11)$$

Then the limit cycle of period q of the model Log-Logistic AR (1) is orbitally stable if and only if

$$\left|\prod_{i=1}^{q}\,[\,\varphi_1 + \pi_1[\frac{1+(1+\beta)\left(\frac{x_{t+q-i}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t+q-i}}{\alpha}\right)^{-\beta}\right)^{2}}]\right| < 1$$

The following proposition is generalization of proposition (1) to Log-Logistic AR (p) model.

2.2. Proposition 2

A limit cycle of period q and q>1 for model Log-Logistic AR(p) is orbitally stable if all the eigen values of the matrix,

$B = B_q.B_{q-1}.\dots.B_1$

Possess absolute value is less than one, where

$$B_i = \begin{bmatrix} b_{1,1}^{(i)} & b_{1,2}^{(i)} & \cdots & b_{1,p-1}^{(i)} & b_{1,p}^{(i)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad i=1,2,3,\dots,q$$

Where

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$$b_{1,1}^{(i)} = \varphi_1 + \frac{\pi_1}{\left(1+\left(\frac{x_{t+i-1}}{\alpha}\right)^{-\beta}\right)^2} + \sum_{j=1}^{p-1} \frac{\pi_j(1+\beta)\left(\frac{x_{t+i-j}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t+i-j}}{\alpha}\right)^{-\beta}\right)\left(1+\left(\frac{x_{t+i-j}}{\alpha}\right)^{-\beta}\right)} \quad \ldots (12)$$

$$b_{1,k}^{(i)} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_{t+i-k}}{\alpha}\right)^{-\beta}\right)^2}; \qquad k = 2,3,\ldots,p \quad \ldots (13)$$

2.2.1. Proof 2

The Log-Logistic AR (p) can be written after canceling the effect of white noise (z_t) in the state space form as

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ \vdots \\ x_{t-p+1} \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p-1} & m_{1,p} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ \vdots \\ x_{t-p} \end{bmatrix} \quad \ldots (14)$$

Where

$$m_{1,k} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2} \; ; \; k = 1,2,3,\ldots,p$$

Let the limit cycle with cycle q and q> 1 be the closed and isolated path known by equation (2.8)

By taking an open neighborhood for each point of the limit cycle, that is by replacing each of x_s with $x_s + \xi_s$ for s = t, t-1, t-2, … , t-q

such that $\xi_s^n \rightarrow 0$ for $n \geq 2$

therefor the equation (14) becomes the formula

$$x_t + \xi_t = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p-1} & m_{1,p} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} (x_{t-1} + \xi_{t-1}) \quad \ldots (15)$$

So that x_t , x_{t-1} , ξ_t ,ξ_{t-1} defined on the state space R^p

$$m_{1,k} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2} \; ; \; k = 1,2,3,\ldots,p$$

and by conducting an approximation of the (first) and (second) terms of Taylor expansion for each element of the matrix and performing simple algebraic operations, we get

$$\xi_t = B_0\, \xi_{t-1} \quad \ldots (16)$$

So ξ_t , ξ_{t+1} defined on the state space R^p and B_0 it has the following formula

$$B_0 = \begin{bmatrix} b_{1,1}^{(0)} & b_{1,2}^{(0)} & \cdots & b_{1,p-1}^{(0)} & b_{1,p}^{(0)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$b_{1,1}^{(0)} = \varphi_1 + \frac{\pi_1}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2} + \frac{\pi_1(1+\beta)\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2}$$

$$b_{1,k}^{(0)} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_{t-1}}{\alpha}\right)^{-\beta}\right)^2} \; ; \; k = 2,3,\ldots,p$$

repeated the relationship (16) again, we get

$$\xi_{t+1} = B_1\, \xi_t \quad \ldots (17)$$

So

$$B_1 = \begin{bmatrix} b_{1,1}^{(1)} & b_{1,2}^{(1)} & \cdots & b_{1,p-1}^{(1)} & b_{1,p}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Where

$$b_{1,1}^{(0)} = \varphi_1 + \frac{\pi_1}{\left(1+\left(\frac{x_t}{\alpha}\right)^{-\beta}\right)^2} + \frac{\pi_1(1+\beta)\left(\frac{x_t}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_t}{\alpha}\right)^{-\beta}\right)^2}$$

$$b_{1,k}^{(0)} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_t}{\alpha}\right)^{-\beta}\right)^2} \; ; \; k = 2,3,\ldots,p$$

repeated the relationship (17) q-times, we get

$$\xi_{t+q} = B_q\, \xi_{t+q-1} = B_q B_{q-1} \xi_{t+q-2} \ldots$$

$$\xi_{t+q} = \prod_{i=1}^{q} B_i \cdot \xi_t \quad \ldots (18)$$

let B be the product of the matrices B_i for $i = 1,2,3,\ldots q$ the equation (18) can be WRITTEN

$$\xi_{t+q} = B \cdot \xi_t \quad \ldots (19)$$

for convergence of B_j to zero as j→∞ then the absolute values of eigenvalues of matrix B must be less than 1. In other words, if the absolute values of the eigenvalues of matrix B lies inside the unite circle

where $B = \prod_{i=1}^{q} B_i$ and

$$B_i = \begin{bmatrix} b_{1,1}^{(i)} & b_{1,2}^{(i)} & \cdots & b_{1,p-1}^{(i)} & b_{1,p}^{(i)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad , i=1, 2, q$$

were

$$b_{1,1}^{(i)} = \varphi_1 + \frac{\pi_1}{\left(1+\left(\frac{x_{t+i-1}}{\alpha}\right)^{-\beta}\right)^2} +$$

$$\sum_{j=1}^{p} \frac{\pi_j(1+\beta)\left(\frac{x_{t+i-j}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{x_{t+i-j}}{\alpha}\right)^{-\beta}\right)^2}$$

$$b_{1,k}^{(i)} = \varphi_k + \frac{\pi_k}{\left(1+\left(\frac{x_{t+i-k}}{\alpha}\right)^{-\beta}\right)^2} \quad ; \; k = 2,3,\dots,p$$

Then the model Log-Logistic AR (p) is orbitally stability if all roots of eigen values of matrix B inside unite circle.

3. APPLICATION

Some examples of stable and unstable limit cycles in this paragraph to illustrate how the conditions for the stability of the limit cycle in proposition (1) are applied. We will use MATLAB program to get the period q, look at appendix.

3.1. Example 1

Let the estimated values $\varphi_1 = -1.3$, $\pi_1 = 2.75$, $\alpha = 1.2$, $\beta = 2$ therefore the Log-Logistic AR (1) model is given by:

$$x_t = \left[-1.3 + \frac{2.75}{1+\left(\frac{x_{t-1}}{1.2}\right)^{-2}}\right] x_{t-1} + z_t$$

By suppressing the white noise process since $(z_t) = 0$, and by use MATLAB program we get a limit cycle of period 2 which is $\{0.419 \; , -0.419\}$ look at appendix. We can calculate it by using the following condition of the equation (3).

$$\left| \prod_{i=1}^{2} \left[-1.3 + 2.75 \left[\frac{1+(1+2)\left(\frac{x_{t+q-i}}{1.2}\right)^{-2}}{\left(1+\left(\frac{x_{t+q-i}}{1.2}\right)^{-2}\right)^2} \right] \right| =$$

$$0.2194 < 1$$

Since the condition of equation (3) is satisfied, the limit cycle is orbitally stable. The stability of a limit cycle with different initial values is displayed in Figure (2).

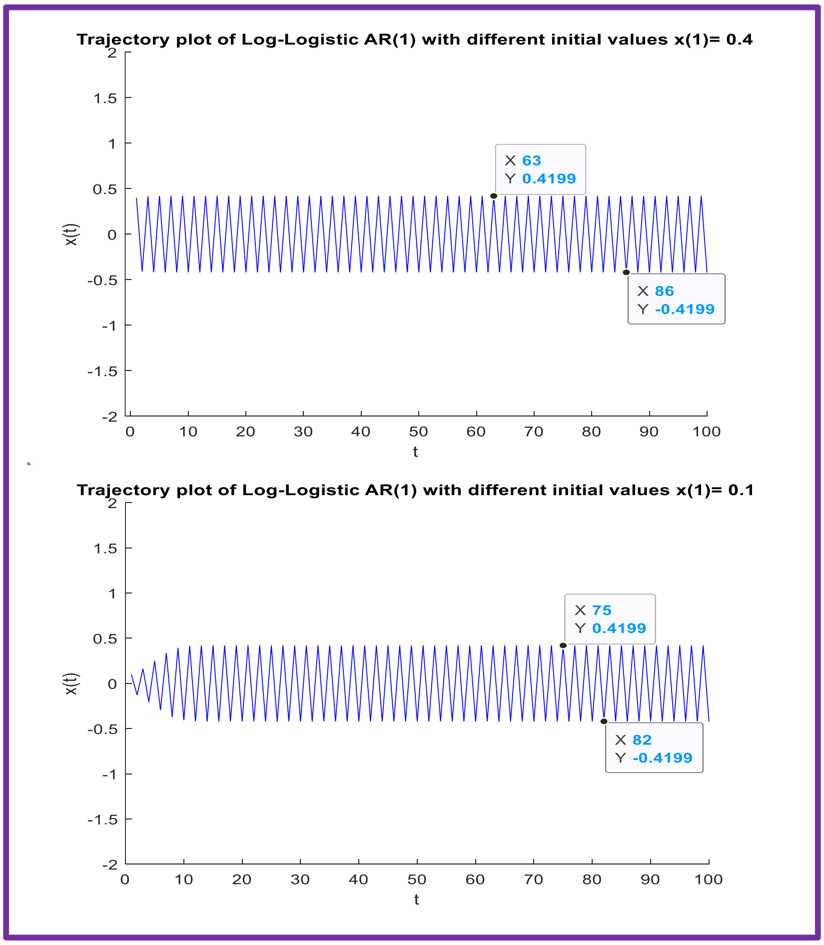

Fig. 2: Illustrates the orbitally stable for different initial values.

3.2. Example 2

Consider the estimated values $\varphi_1 = 2.7$, $\pi_1 = -3.6$, $\alpha = 2.45$, $\beta = 3$ therefore the the Log-Logistic AR (1) model is given by

$$x_t = \left[2.7 - \frac{3.6}{1+(\frac{x_{t-1}}{2.45})^{-3}} \right] x_{t-1} + z_t$$

By suppressing the white noise process since $z_t = 0$, and by use MATLAB program we get a limit cycle of period 4 that is {2.9 , 1.2 , 2.8 , 1.3 , 2.9} , and we can calculate that by using the following conditions of equation (3)

$$\left| \prod_{i=1}^{4} [2.7 - 3.6[\frac{1+(1+3)\left(\frac{x_{t+q-i}}{2.45}\right)^{-3}}{\left(1+\left(\frac{x_{t+q-i}}{2.45}\right)^{-3}\right)^2}]] \right| = 5.6272 > 1$$

The limit cycle is orbitally unstable since does not satisfy the condition of equation (3). Figure (3) illustrates the limit cycle unstable with the different initial values.

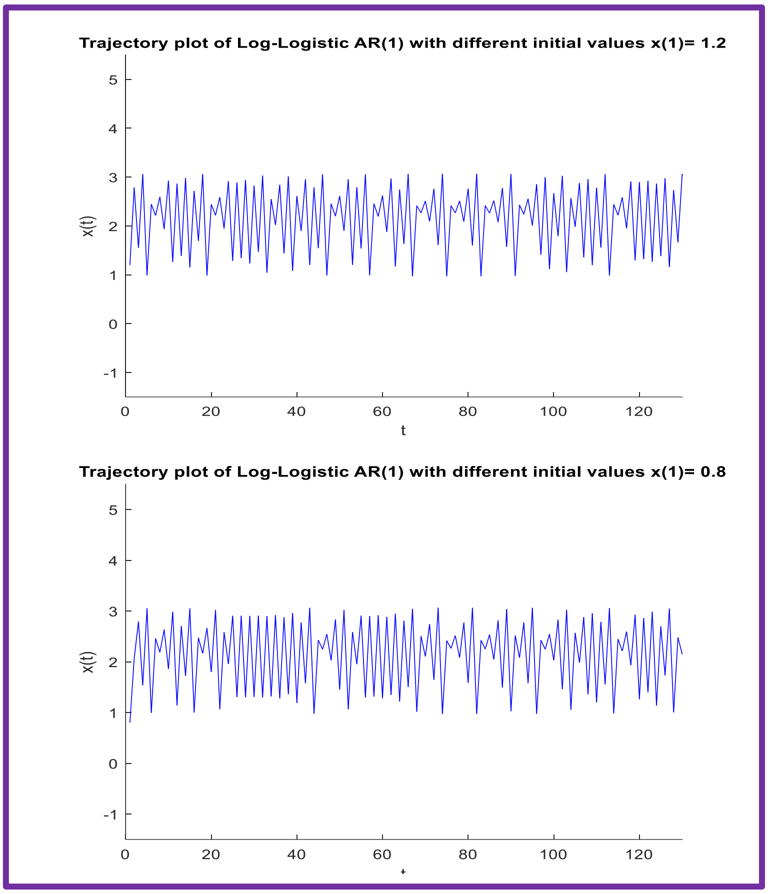

Fig. 3: Illustrates the orbitally unstable for different initial values.

4. CONCLUSION

In this study, we found and prove the stability conditions of the limit cycle for the Log-Logistic autoregressive model by using two proposition, in proposition (1), we used the local linearization approximation technique to find the stability conditions of the limit cycle for the period (q > 1). And we give two examples by taking estimation values and used MATLAB program to find the periods with their plots to explain if the Log-logistic AR (1) is an orbital stable or not. Furthermore, proposition (2) is a generalization of proposition (1) to (Log-Logistic AR) of order p, which used the state space to analyze the stability conditions of the limit cycle.

Appendix

In below algorithm of MATLAB program:

```
clear; clc;
a1=-1.3; b1=2.75; c=1.2; d=2; a=a1; b=b1;
k=b/(1-a); K=zeros (1,200);
xlabel('t'), ylabel('x(t)');
```

```
title ('Trajectory plot of Log-Logistic AR (1) with different initial values x (1) = 0.1');
axis ([0 100 -2 2]);
N=408; number. data=N;
for j=1:1
  K(j)=0.1;
end
for i=2:200
 K(i)=a1*K(i-1)+(b1*K(i-1))*(1/(1+(K(i-1)/c)^(-d)));
    hold on
plot([(i), (i-1)],[K(i),K(i-1)],'b');
   hold off
end
disp(K);
```

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