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Using Kharrat-Toma transform to solve a growth and decay problem

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ABSTRACT

Many academics have recently concentrated on applying integral transforms to solve challenging problems in economics, biology, engineering, and medicine. In this paper, the Kharrat-Toma transform is used to issues related to population growth and decay problem. These are important topics in the fields of finance, physics, chemistry, biology, and society. Many applications show the value and efficacy the Kharrat-Toma transform is used to analyze concerns with population growth and decrease. The findings demonstrate the usefulness of the Kharrat-Toma transform in resolving problems associated with population growth and deterioration. In population growth and decay problems there is a differential equation model in mathematics that can be analyzed by the Kharat-Toma transform. We have shown its analysis and applications in this article.

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تطبيق تحويل خارات-توما لحل مشاكل النمو السكاني والاضمحلال

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الملخص

وقد ركز العديد من الأكاديميين مؤخرًا على تطبيق التحويلات المتكاملة لحل المشكلات الصعبة في الاقتصاد والبيولوجيا والهندسة والطب. في هذا البحث، تم استخدام تحويل خارات-توما لمشاكل النمو السكاني والاضمحلال. هذه موضوعات مهمة في مجالات التمويل والفيزياء والكيمياء والبيولوجيا والمجتمع. تُظهر العديد من التطبيقات قيمة وفعالية تحويل خراط-توما في قضايا النمو السكاني والانحطاط. وتبين النتائج فائدة تحويل خارات – توما في حل المشكلات المرتبطة بالنمو والتدهور السكاني.

1- Introduction

Mathematical models are crucial for improving how accurate the biotechnological process is, and differential equations are often employed in many branches of science and engineering to represent complicated physical events [1, 2]. Integral transforms are currently regarded as among the most practical and straightforward mathematical techniques to find solutions to complex problems in a variety of fields such as science, technology, engineering, and finance. The potential of integral transforms to give exact solutions to problems without requiring lengthy calculations is a crucial feature and advantage.

Assessment of the improper integrals by utilising the Sawi transform of error functions. In [3] author explored the first kind of Volterra Integral equations solution with utilizing transform of Emad-Sara. Then and in [4] utilized transform of Anuj to solve the first kind Volterra Integral equations. System of differential equations solved by see [5] using Soham transform [6] by utilized the transform of Emad-Sara to solve telegraph equation.

Using of HY integral transform to deal with problems of growth and decay. Transform of Emad-Falih utilized [7] for general solution of telegraph equation. Transform of Double kushare was explored and investigated [8]. Currently, Emad Sara, Emad-Falih and Alenzi transform have been used [9-11] that also used to solve multi-higher order fractional differential equations by many authors in [12]. to solve Issues related to population growth and decline. Rishi transform introduced in [13].

There are many applications of modeling in real-life problems, and in mathematical fields such as fractional calculus, modeling is much more detailed, such as physics, chemistry, and mechanics [14-18].

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In our study, the Kharrat-Toma transform is used to solve problems of growth and decay. The paper is organized as follows: Section two contains preliminaries. In Section three Kharrat-Toma transform is utilized for problems of growth and decay. Applications are drawn in Section four and the last section gives the conclusion.

2- Preliminaries

Basic properties of Kharrat-Toma transform.

In this section, we have established the Kharrat-Toma transform and elucidated its characteristics. Through a number of additional studies, we shall gain an overview of the new transform.

Definition 1 [Kharrat-Toma transforms, [19]]. The exponentially ordered piecewise continuous function with the Kharrat-Toma transform the function $f(t)$ defined in the interval $[0,\infty)$, is expressed as:

$B[f(t)] = \delta^3 \int_0^\infty f(t)e^{\frac{-t}{\delta^2}}dt = F(\delta)\,,\ t \geq 0, \delta \in R$

Property 1: [19], [20] Some basic functions and their Kharrat-Toma transform $t \geq 0, \delta \in R$.

$(t), t > 0$	$B\{f(t)\} = F(\delta)$	$f(t), t > 0$	$B\{f(t)\} = F(\delta)$
1	δ^5	$sin(at)$	$\frac{a\delta^7}{1+a^2\delta^4}$
e^{kt}	$\frac{\delta^5}{1-k\delta^2}$	$cos(at)$	$\frac{\delta^5}{1+a^2\delta^4}$
$t^n, n \in N$	$n!\,\delta^{2n+5}$	$sinh(at)$	$\frac{a\delta^7}{1-a^2\delta^4}$
$t^\beta, \beta > -1, \beta \in R$	$\Gamma(\beta+1)\delta^{2\beta+5}$	$cosh(at)$	$\frac{\delta^5}{1-a^2\delta^4}$

Property 2: [19], [20] Inverse Kharrat-Toma transformations for some fundamental functions.

$F(\delta)$	$B^{-1}\{F(\delta)\} = f(t)$	$F(\delta)$	$B^{-1}\{F(\delta)\} = f(t)$
δ^5	1	$\frac{a\delta^7}{1+a^2\delta^4}$	$sin(at)$
$\frac{\delta^5}{1-k\delta^2}$	e^{kt}	$\frac{\delta^5}{1+a^2\delta^4}$	$cos(at)$
$n!\,\delta^{2n+5}$	$t^n, n \in N$	$\frac{a\delta^7}{1-a^2\delta^4}$	$sinh(at)$
$\Gamma(\beta+1)\delta^{2\beta+5}$	$t^\beta, \beta > -1, \beta \in R$	$\frac{\delta^5}{1-a^2\delta^4}$	$cosh(at)$

Property 3 :[19] **(Convolution Theorem)** let $A\{k(x)\} = K(\delta)$ and $A\{l(x)\} = L(\delta)$ then

$B[k(t) * l(t)] = \frac{1}{\delta^2}K(\delta)L(\delta)$

Such that $*$ shows the convolution of L and l , then $k(x) * l(x) = \int_0^u k(x-u)l(u)du.$

Property 4: ([19], [20]). Both the Kharrat-Toma transform and its inverse are linear operators.

1. Kharrat-Toma Transformation is linear operator

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$B\{\sum_{i=0}^{n} a_i f_i(t)\} = \sum_{i=0}^{n} a_i B\{f_i(t)\}$. such that a_i are arbitrary constants.

2. Inverse Kharrat-Toma Transformation is linear operator

If $f_i(u) = B^{-1}\{F_i(\delta)\}$, then $B^{-1}\{\sum_{i=0}^{n} a_i F_i(\delta)\} = \sum_{i=0}^{n} a_i B^{-1}\{F_i(\delta)\}$, such that a_i are arbitrary constants.

Property 5: (see[19], [20]). The Kharrat-Toma transformation for integer order derivative of $f(u)$ is:

$$B[f^n(t)] = \frac{1}{\delta^{2n}} F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0) \ldots (1).$$

3. Population growth model

A first order ordinary linear differential equation can be used to mathematically describe the population expansion of a plant, cell, organ, or species. [21], [22] as $\frac{dy}{dt} = ky \ldots .(2)$

with initial condition $y(t_0) = y_0$ such that $k \in R^+$, Let y represent the population size at time u, and y_0 represent the original population size at time $t = t_0$.

The above differential equation is mathematical model for rate of growth.

The Malthusian law of population increase is referred to as Equation (2).

The decay problem of the material may be quantitatively described by a first-order ordinary linear differential equation [21], [22] as $\frac{dy}{dt} = -ky \ldots (3)$

with initial condition $y(t_0) = y_0$

Where y denotes the substances quantity at time t, k is a positive real number, and y_0 denotes the substances initial quantity at time $t = t_0$.

The negative sign in equation (3) is used on the right-hand side to account for the decreasing mass of the material with time. Consequently, the derivative $\frac{dy}{dt}$ must be negative.

3.1. Kharrat-Toma transform for population growth problem

We introduce the Kharrat-Toma transform for the population increase problem in this part. Its mathematical formulation is presented by (2)

Taking Kharrat-Toma transform on both sides of (2), we have $B\left\{\frac{dy}{dt}\right\} = kB\{y(t)\}$

Applying the Kharrat-Toma transform of the derivative of a function, property 5, to equation (2), we obtain:

$B[f^n(t)] = \frac{1}{\delta^{2n}} F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0)$, we have $n = 1$

So, we obtain $\frac{1}{\delta^2} Y(\delta) - \sum_{k=0}^{0} \delta^{3+2k} y^k(0) = Bk\{y(t)\}$

$$\frac{1}{\delta^2} Y(\delta) - \delta^3 y(0) = kY(\partial) \ldots .(4)$$

Using initial condition $y(t_0) = y_0$ in (4) also about simplicity, we have

$$\left(\frac{1}{\delta^2} - k\right) Y(\delta) = \delta^3 y(0) \quad , \quad \Rightarrow Y(\delta) = \frac{\delta^5 y(0)}{1 - \delta^2 k} \ldots .(5)$$

Operating inverse Kharrat-Toma transform on both sides of (5), we obtain

$$B^{-1}\{Y(\delta)\} = B^{-1}\left\{\frac{\delta^5 y(0)}{1 - \delta^2 k}\right\} \Rightarrow y(t) = B^{-1}\left\{\frac{\delta^5 y(0)}{1 - \delta^2 k}\right\} \Rightarrow y(t) = y_0 e^{kt} \ldots (6)$$

which, at time ***u***, is the necessary population number.

3.2. KHARRAT-TOMA TRANSFORM FOR DECAY PROBLEM

In this section, we present Kharrat-Toma transform for decay problem which is mathematically expressed in terms of (3).

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Applying the Kharrat-Toma transform on both sides of (3), we have

$B\left\{\frac{dy}{dt}\right\} = -kB\{y(t)\}$ ... (7)

The above differential equation is mathematical model for rate of growth.

Now applying the property 5, Kharrat-Toma transforms of derivative of function, on (7), we have $B[f^n(t)] = \frac{1}{\delta^{2n}}F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0)$, we have $n = 1$

So, we obtain

$\frac{1}{\delta^2}Y(\delta) - \sum_{k=0}^{0} \delta^{3+2k}y^k(0) = -Bk\{y(t)\} \rightarrow$

$\frac{1}{\delta^2}Y(\delta) - \delta^3 y(0) = -kY(\partial)$ ... (8),

Using initial condition $y(t_0) = y_0$ in (8) also about simplicity, we have

$\left(\frac{1}{\delta^2} + k\right)Y(\delta) = \delta^3 y(0)$, $\Rightarrow Y(\delta) = \frac{\delta^5 y(0)}{1+\delta^2 k}$ ..(9)

Operating inverse Kharrat-Toma transform on both sides of (9), we obtain

$B^{-1}\{Y(\delta)\} = B^{-1}\left\{\frac{\delta^5 y(0)}{1+\delta^2 k}\right\} \Rightarrow y(t) = B^{-1}\left\{\frac{\delta^5 y(0)}{1+\delta^2 k}\right\} \Rightarrow y(t) = y_0 e^{-kt}$ ..(10)

which, at time ***u***, is the necessary population number.

4- Applications

The usefulness of the Kharrat-Toma transform for population growth and decay issues is illustrated in a few cases provided in this section.

Application 1: The pace of population growth in a Dukan is directly proportional to the total number of inhabitants already living there. Calculate how many people were originally residing in the city if, after ten years, the population has triple and, after 12 years, it is 20,000.

Solution: This problem can be written in mathematical form as: $\frac{dy}{dt} = ky$ ... (11)

where y denote the number of people living in the city at any time t and k is the constant of proportionality. Consider y_0 is the number of people initially living in the city at $t = 0$.

Now applying the property 5, Kharrat-Toma transforms of derivative of function, on (11), we have

$B[f^n(t)] = \frac{1}{\delta^{2n}}F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0)$, we have $n = 1$

So, we obtain $\frac{1}{\delta^2}Y(\delta) - \sum_{k=0}^{0} \delta^{3+2k}y^k(0) = Bk\{y(t)\}$

$\frac{1}{\delta^2}Y(\delta) - \delta^3 y(0) = kY(\partial)$

Using initial condition $y(t_0) = y_0$ in above equation and on simplification, we have

$\left(\frac{1}{\delta^2} - k\right)Y(\delta) = \delta^3 y(0)$, $\Rightarrow Y(\delta) = \frac{\delta^5 y(0)}{1-\delta^2 k}$

Operating inverse Kharrat-Toma transform on both sides of above equation, we obtain

$B^{-1}\{Y(\delta)\} = B^{-1}\left\{\frac{\delta^5 y(0)}{1-\delta^2 k}\right\} \Rightarrow y(t) = B^{-1}\left\{\frac{\delta^5 y(0)}{1-\delta^2 k}\right\} \Rightarrow y(t) = y_0 e^{kt}$ ... (12)

Now at $t = 10, y = 3y_0$, so using this in (12), we have $3y_0 = y_0 e^{10k} \Rightarrow e^{10k} = 3$

$\Rightarrow k = 0.1ln3 = 0.1098$ ... (13),

Now using the condition at $t = 12, y = 20{,}000$, in (12), we have

$20{,}000 = y_0 e^{12k}$ ... (14),

putting the value of k from (13) in (14), we have

$20{,}000 = y_0 e^{12\times 0.1098} \Rightarrow 20{,}000 = 3.7344 y_0 \Rightarrow y_0 \cong 5355\ person$

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which are the required number of people initially living in the Dukan.

In figure 1 the growth rate and exact solution of application 1 have been presented.

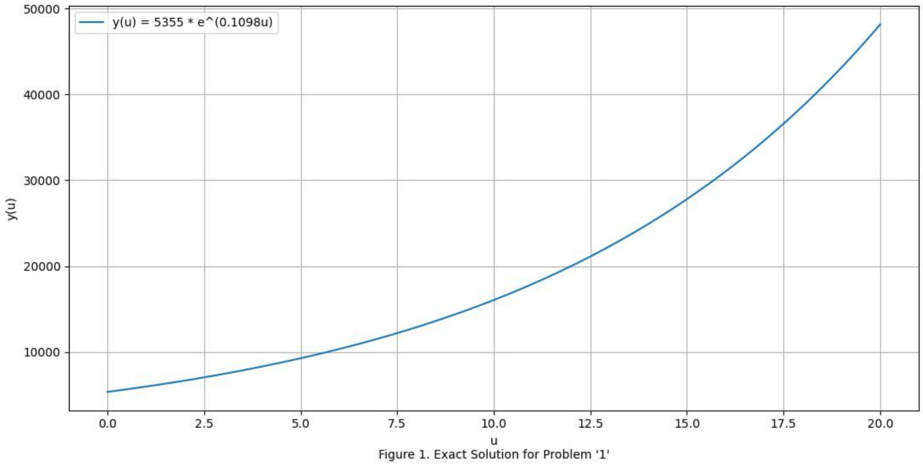

Fig. 1

Application 2: It is well known that the rate at which a radioactive material decays is related to its concentration. Determine the radioactive substance's half-life if there are 160 milligrams of it originally and it is seen that after four hours, the material has lost twenty percent of its initial mass.

Solutions: This issue can be expressed mathematically as: $\frac{dy(t)}{dt} = -ky(t)$ ... (15)

where y is the amount of radioactive material at time u and k is the proportionality constant. At time $t = 0$, let $y_0$represent the initial amount of radioactive material. By applying the Kharrat-Toma transform of the function's derivative on (15) and the property 5, we now have

$B[f^n(t)] = \frac{1}{\delta^{2n}} F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0)$, we have $n = 1$So, we obtain

$\frac{1}{\delta^2} Y(\delta) - \sum_{k=0}^{0} \delta^{3+2k} y^k(0) = -Bk\{y(t)\} \to$

$\frac{1}{\delta^2} Y(\delta) - \delta^3 y(0) = -kY(\partial)$,

Using initial condition $y(t_0) = y_0$ in above equation and on simplification, we have

$\left(\frac{1}{\delta^2} + k\right) Y(\delta) = \delta^3 y(0)$, $\Rightarrow Y(\delta) = \frac{\delta^5 y(0)}{1+\delta^2 k}$

With the inverse Kharrat-Toma transform applied to both sides of the equation above, we get

$B^{-1}\{Y(\delta)\} = B^{-1}\left\{\frac{\delta^5 y(0)}{1+\delta^2 k}\right\} \Rightarrow y(t) =$

$B^{-1}\left\{\frac{\delta^5 y(0)}{1+\delta^2 k}\right\} \Rightarrow y(t) = y_0 e^{-kt}$ ... (16)

Since the radioactive material has now lost 20% of its initial mass of 160mg at $t = 4$,

$y = 100 - 20 = 80$. Applying this to equation (16), we have

$80 = 160e^{-4k} \Rightarrow e^{-4k} = 0.5 \Rightarrow k =$

$-0.25\, ln\, 0.5 = 0.173$ .... (17)

We required t when $y = \frac{y_0}{2} = \frac{160}{2} = 80$ so from (17), we get $80 = 160e^{-kt}$ ... (18)

Putting the value of k from (17) in (18), we have

$80 = 160e^{-0.173t} \Rightarrow e^{-0.173t} = 0.5 \Rightarrow t =$

$-\frac{1}{0.173} ln 0.5 \Rightarrow t \cong 4$ hours.

Which is the radioactive substances necessary half-life

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In figure 2 the growth rate and exact solution of application 2 have been presented

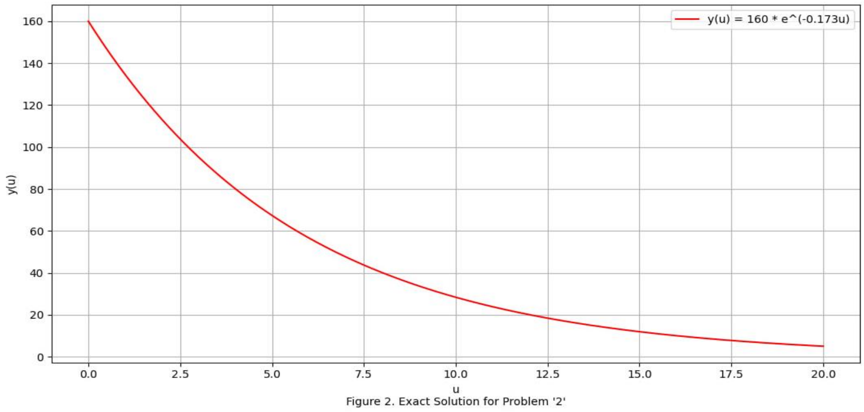

Fig. 2

Application 3: A culture initially has y_0 number of bacteria. At $t=1\,h$ the number of bacteria is measured to be $\frac{3}{2}y_0$. Find the amount of time required for the number of bacteria to triple if the rate of growth is proportionate to the number of bacteria $y(t)$ present at time t.

Solution: This problem can be written in mathematical form as: $\frac{dy}{dt}=ky$ ... (11)

where y denote the number of bacteria in the culture at any time t and k is the constant of proportionality. Consider y_0 is the number of bacteria initially in the culture at $t=0$.

Now applying the property 5, Kharrat-Toma transforms of derivative of function, on (11), we have

$B[f^n(t)] = \frac{1}{\delta^{2n}}F(\delta) - \sum_{k=0}^{n-1} \delta^{-2n+2k+5} f^k(0)$, we have $n=1$

So, we obtain $\frac{1}{\delta^2}Y(\delta) - \sum_{k=0}^{0} \delta^{3+2k} y^k(0) = Bk\{y(t)\}$

$\frac{1}{\delta^2}Y(\delta) - \delta^3 y(0) = kY(\partial)$

Using initial condition $y(t_0)=y_0$ in above equation and on simplification, we have

$\left(\frac{1}{\delta^2}-k\right)Y(\delta) = \delta^3 y(0)$, $\Rightarrow Y(\delta) = \frac{\delta^5 y(0)}{1-\delta^2 k}$

Operating inverse Kharrat-Toma transform on both sides of above equation, we obtain

$B^{-1}\{Y(\delta)\} = B^{-1}\left\{\frac{\delta^5 y(0)}{1-\delta^2 k}\right\} \Rightarrow y(t) = B^{-1}\left\{\frac{\delta^5 y(0)}{1-\delta^2 k}\right\} \Rightarrow y(t) = y_0 e^{kt}$ .... (12)

Now at $t=0, y=y_0$, and at $t=1, y=\frac{3}{2}y_0$ so using this in (12), we have We first solve the differential equation in (1), with the symbol x replaced by y. With $t_0=0$ the initial condition is $y(0)=y_0$. We then use the empirical observation that $y(1)=\frac{3}{2}y_0$ to determine the constant of proportionality k.

Therefore $y(t)=ce^{kt}$. At $t=0$ it follows that $y_0=ce^0=c$, so $y(t)=y_0e^{kt}$. At $t=1$ we have $\frac{3}{2}y_0=y_0e^k$ or $e^k=\frac{3}{2}$. From the last equation we get $k=ln\frac{3}{2}=0.4055$, so $y(t)=y_0e^{0.4055t}$. To find the time at which the number of bacteria has tripled, we solve $3y_0=y_0e^{0.4055t}$ for t. It follows that $0.4055t=ln3$, or

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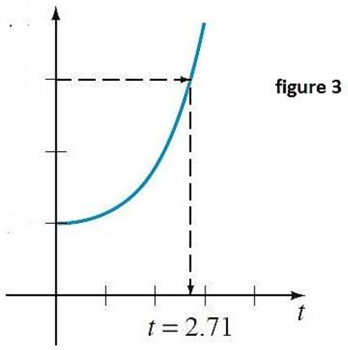
$t = \frac{\ln 3}{0.4055} \approx 2.71\ h.$

In figure 3 the growth rate and exact solution of application 3 have been presented



5- Result and discussion

After using the Kharat-Toma transformation and the applications on which we have implemented the method, it turned out that this transformation analyzes the growth and decay models without complications and provides exact solutions.

6- Conclusion

The population growth and decay issues have been successfully resolved in this work by developing the Kharrat-Toma transform. The applications provided demonstrate how well the Kharrat-Toma transform works to address issues with population degradation and expansion. The study's findings show that the suggested transform generates precise answers without the need for a difficult computation. The Kharrat-Toma transform may be used in the future to address a variety of challenging issues in science, technology, and medicine. by developing models in mathematics. If In research writing, conclude our results, we have come to the conclusion that It turned out that the Kharat-Toma transformation analyzes the growth and decay models without difficulty and gives precise solutions after utilizing it on the applications on which we have applied the method.

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9- Author contribution: Authors contributed equally in the study.

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