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Modelling the Impact of Vaccination and Quarantine in the Dynamics of Mumps Infection with Hearing Loss

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ABSTRACT

Hearing loss is a growing public health concern with serious implications for an individual's quality of life. This paper proposes a mathematical model that describes the dynamics of the spread of mumps and the associated risk of hearing loss caused by mumps infection. The model incorporates key epidemiological factors and considers the role of vaccination and quarantine as control measures. Mathematical analysis of the model was carried out to ensure positivity and boundedness of solutions over time. The model exhibits two steady states: a mumps-free steady state and a mumps-endemic steady state. Stability and sensitivity analyses show the effectiveness of quarantine and administering vaccine in minimizing the spread of mumps and consequently mitigating hearing loss. Though, quarantine has high significant impact on the dynamics with first dose of vaccination, the outcome after the second dose of vaccination is far better. In addition, the findings emphasize the importance of certain parameters in shaping the disease dynamics and offer guidance on effective intervention strategies. The study underscores the role of vaccination and quarantine in mitigating the impact of mumps-induced hearing loss.

Keywords: Hearing loss, Mumps, reproduction number, mathematical model, stability analysis.

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النموذج الرياضي لفقدان السمع الناتج عن النكاف

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الملخص

فقدان السمع هو مصدر قلق متزايد للصحة العامة له آثار خطيرة على جودة حياة الفرد. تقترح هذه الورقة نموذجاً رياضياً يصف ديناميكيات انتشار النكاف والمخاطر المرتبطة به من فقدان السمع الناجم عن عدوى النكاف. يتضمن النموذج عوامل وبائية رئيسية ويأخذ في الاعتبار دور التطعيم والحجر الصحي كإجراءات للسيطرة. تم إجراء تحليل رياضي للنموذج لضمان إيجابية وتقييد الحلول مع مرور الوقت. يُظهر النموذج حالتين مستقرتين: حالة مستقرة خالية من النكاف وحالة مستقرة متوطنة بالنكاف. تُظهر تحليلات الاستقرار والحساسية فعالية الحجر الصحي وإعطاء اللقاح في تقليل انتشار النكاف وبالتالي التخفيف من فقدان السمع. على الرغم من أن الحجر الصحي له تأثير كبير على الديناميكيات مع الجرعة الأولى من التطعيم، إلا أن النتيجة بعد الجرعة الثانية من التطعيم أفضل بكثير. بالإضافة إلى ذلك، تؤكد النتائج على أهمية بعض المعايير في تشكيل ديناميكيات المرض وتقدم إرشادات حول استراتيجيات التدخل الفعالة. تؤكد الدراسة على دور التطعيم والحجر الصحي في التخفيف من تأثير فقدان السمع الناجم عن النكاف.

INTRODUCTION

Mumps is an infection of the salivary glands caused by a virus belonging to the paramyxovirus group. This virus is an RNA virus that spreads through direct contact and droplets. Mumps typically affects children and can lead to acute respiratory infections. Early symptoms include fever, muscle pain, headaches, and fatigue. Mumps is usually followed by painful swelling of one or both parotid glands [(1)]. According to [(2)], mumps symptoms in adults are often more severe than in children. Severe symptoms can result in complications such as aseptic meningitis, encephalitis, orchitis, oophoritis, and, in severe cases, infections of the brain cover (15%), pancreatitis (4%), permanent deafness, and painful testicular swelling that rarely causes infertility. However, individuals who recover from this viral infection usually acquire lifelong immunity, and the chances of reinfection are generally low [(2, 3)].

Prevention of mumps spread is achieved through the MMR (Measles, Mumps, Rubella) vaccine, administered to children at ages $12-15$ months for the first dose and $4-6$ years for the second dose. However, the MMR vaccine has limited effectiveness in controlling mumps transmission [(1)]. Additionally, ongoing genetic mutations of the virus and individual movement from one region to another contribute to the spread, leading to epidemics in certain areas. An example is the situation in Xiamen city and Fujian province, China, where mumps affects around 21 individuals out of 100 in the total population. Mumps epidemics are common throughout the year near the equator, while in regions further north and south of the equator, they often occur during winter and spring, resulting in approximately one in ten thousand infected individuals dying [(4-6)]. This makes mumps a dangerous disease that can cause mass fatalities if

its spread is not effectively controlled. Hence, several studies have been conducted to address mumps spread, including [4], which discusses mathematical modeling of mumps vaccine failure in Jiangsu Province, China. This research divides the infected population into two sub populations, assuming no deaths due to mumps. The findings recommend health program planners to implement more preventive interventions during periods of higher infection risk. Additionally, The study [7] developed a dynamic transmission model to assess the cost-effectiveness of routine one- and two-dose mumps vaccination programs in Japan. Both programs were found to be cost-effective and saved quality-adjusted life years (QALYs) compared to the current program. The two-dose vaccination programs was consistently more cost-effective and QALY-saving than the one-dose program throughout the study period, confirming its superiority. This analysis provides valuable insights for policy decisions regarding mumps vaccination not only in Japan but also in other countries where the mumps vaccine is not part of the national immunization program.

Researchers in [8] discusses seasonal mumps spread in China, defining classes of individuals vaccinated with MMR. The analysis suggests that vaccination rates and vaccine effectiveness play crucial roles in mumps spread. The study proposes increasing vaccine coverage and implementing two doses of MMR vaccine in Mainland China. Another study by [4] explores the correlation between mumps and meteorological factors in Xiamen, China. It distinguishes classes of individuals exhibiting mild and severe infection symptoms. The conclusion is that a relatively high transmission rate in Xiamen leads to sustained mumps epidemics. Meteorological factors, especially air temperature and relative humidity, are closely related to mumps. Furthermore, another study [9] uses a density-dependent SEIR model, asserting that mumps cases increase with the city's size, indicating density-dependent transmission. The researchers gathers data from various American cities between 1923-1932, concluding that mumps cases most frequently increase in March, with higher transmission than average from December to April, correlating with weekly births. The results of the study emphasize the importance of long-term infectious disease survey data, providing insights into future studies on mumps resurgence [9]. Another study [6] examined the spread of mumps by using the SIQR model dividing the population into four classes and assuming that deaths from mumps were caused by complications with other diseases. There is also a quarantine class for mumps-infected individuals.

Some Mathematical models have been used to study the dynamics of hearing loss caused by hazardous exposure to noise and hearing loss can also be caused by hereditary factors, birth complicates, certain viral diseases, chronic ear infections, the use of certain medicines, excessive noise exposure, and aging, which is a major worldwide health issue [10, 11]. According to the World Health Organization (WHO), more than 430 million people, or 5% of the world's population, require rehabilitation to address their "disabling" hearing loss and 34 million of these are children. By 2050, it is estimated that this number will increase to more than 700 million individuals or one out of every 10 people [12].

Hearing loss is linked to a variety of infections and viral diseases, such as mumps. Mathematical models have been formulated to analysed the function and dysfunction of the inner ear by using partial differential equation while other researchers used SIR model to describe the dynamics of hearing loss caused by viral Infection such as mumps (contagion factor) and social factors [10, 11]. The research in [13]analyzes the consequences of mumps infection, specifically hearing loss, using the Caputo-Fabrizio fractional model. Additionally, some researchers in [14] proposes an SLR model, with L representing hearing impairment due to mumps virus. Their model aim to investigate the

impact of noise on mumps virus growth using Fourier transformation. They conclude that diffusive effects on the system can influence better infection control, intensity, spread, and treatment, leading to improved recovery rates.

Recent studies highlight the persistent challenge of mumps outbreaks despite vaccination efforts, particularly among children and young adults in China. Epidemiological analyses have shown that a significant proportion of mumps cases occur in vaccinated individuals, emphasizing the need for enhanced immunization strategies and surveillance[15]. Additionally, mathematical models incorporating treatment delays have provided valuable insights into the disease's progression and control. Delayed treatment can significantly impact infection dynamics, influencing stability, periodic outbreaks, and control strategies [16]. These findings underscore the importance of considering delay effects in mathematical modeling to optimize intervention measures and reduce the burden of mumps-related complications.

Based on previous studies, there is no sufficient research yet focusing on mumps spread with quarantined population and the advance effect on infected individuals who are experiencing hearing loss due to the mumps virus. This research studies the spread of mumps by using the $SIQHR$ model with five population classes. Comparing our model with [6], we assume there is no deaths due to mumps and in addition, those who are mildly infected with the virus are not quarantined, but those who are severely infected are quarantined, and a small proportion experienceing hearing loss, especially on one side. Furthermore, our model consider the effect of first and second dose of vaccine and those who recover develops life-long immunity.

MATHEMATICAL MODEL

Model description

In this section, we present a new mathematical model by using ordinary differential equations to comprehend the dynamics of the mumps virus, known to be associated with hearing loss. Our model classifies the population into five compartments: S represents the number of susceptible individuals, I corresponds to the number of individuals infected with the mumps virus, H indicates the number of infected individuals experiencing hearing loss, Q accounts for those individuals under quarantine due to infection, and R quantifies the number of individuals recovering from the mumps virus at time t. The interactions among these five compartments are depicted in Fig. 1.

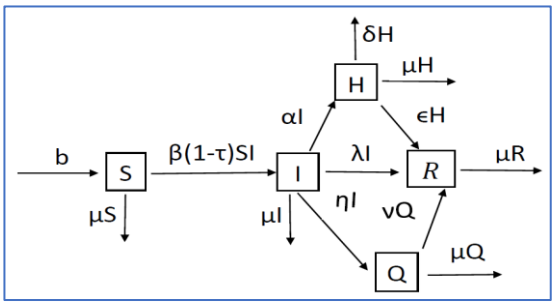

Fig. 1: Diagram of mumps virus.

Below is the equations of the model:

$$\frac{dS}{dt} = b - \beta(1-\tau)S(t)I(t) - \mu S(t)$$

$$\frac{dI}{dt} = \beta(1-\tau)S(t)I(t) - (\alpha + \mu + \lambda + \eta)I(t)$$

$$\frac{dH}{dt} = \alpha I(t) - (\epsilon + \mu + \delta)H(t) \qquad (1)$$

$$\frac{dQ}{dt} = \eta I(t) - (\mu + \nu)Q(t)$$

$$\frac{dR}{dt} = \lambda I(t) + \nu Q(t) + \epsilon H(t) - \mu R(t).$$

With the initial condition: $S(0) > 0, I(0) \geq 0, H(0) \geq 0, Q(0) \geq 0, R(0) \geq 0,$

$$\tau \geq 0, t \in [-\tau, 0]$$

Where $\tau = \tau_1 + \tau_2$.

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Table 1: Description of the model parameters.

Parameter	Parameter's description
b	Recruitment rate of the population
β	The transmission rate of mumps virus
α	Rate of hearing loss as a result of mumps virus
μ	Natural death rate
δ	The rate of death emanating from falls or accidents as a result of hearing loss
ϵ	Recovery rate of mumps virus after it cause hearing loss
λ	Recovery rate
η	Quarantine rate of individuals who have been infected with mumps
ν	Recovery rate of quarantined individuals
τ	Represent administration of vaccinated individuals

Also defining the following equations:

$$N = S + I + H + Q + R, \qquad (2)$$

then the derivative of N is given below with respect to t:

$$\frac{d}{dt}(S + I + H + Q + R) = b - \mu N - \lambda I - \delta H \geq 0,$$

$$\frac{d}{dt}(S + I + H + Q + R) \leq b - \mu N,$$

It follows that:

$$\lim_{\iota \to \infty} Sup(S + I + H + Q + R) \leq \frac{b}{\mu}.$$

Therefore, the feasible region of the system (1) is given by:

$\Gamma = \{(S, I, H, Q, R): S + I + H + Q + R \leq \frac{b}{\mu}, S(0) > 0, I(0) \geq 0, H(0) \geq 0, Q(0) \geq 0, R(0) \geq 0\}$.

Basic reproduction number

It is an effective parameters to predict how different measures affect the way the disease transmits among the population, most of the time denoted by R_0. By using the next generation matrix we discuss the basic reproduction number of the system (1).

$$Let\ x^I = \begin{bmatrix} I \\ H \\ Q \end{bmatrix}, x^N = \begin{bmatrix} S \\ R \end{bmatrix} and \frac{d}{dt}\begin{bmatrix} I \\ H \\ Q \end{bmatrix} = \begin{bmatrix} \beta(1-\tau)SI \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} (\alpha+\mu+\lambda+\eta)I \\ -\alpha I + (\epsilon+\mu+\delta)H \\ -\eta I + (\mu+\nu)Q \end{bmatrix} then\ F(S,I,H,Q,R) = \begin{bmatrix} \beta(1-\tau)SI \\ 0 \\ 0 \end{bmatrix},$$

$$V(S,I,H,Q,R) = \begin{bmatrix} (\alpha+\mu+\lambda+\eta)I \\ -\alpha I + (\epsilon+\mu+\delta)H \\ -\eta I + (\mu+\nu)Q \end{bmatrix},$$

where F and V stands for first and second part of infected compartments.

It is clear that the free disease equilibrium point of the system (1) is equal to $\left(\frac{b}{\mu}, 0,0,0,0\right)$. Now, we define the Jacobian matrices Z and W as follows:

$$Z = \left(\frac{\partial F}{\partial(I,H,Q)}\right)\Big|_{\left(\frac{b}{\mu}0,0,0,0\right)} = \frac{\partial}{\partial(I,H,Q)}\begin{bmatrix} \beta(1-\tau)SI \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \beta(1-\tau)S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\beta(1-\tau)b}{\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$W = \left(\frac{\partial_V}{\partial(I,H,Q)}\right)\Big|_{\left(\frac{b}{\mu}0,0,0,0\right)}$$

$$= \begin{bmatrix} \alpha+\mu+\lambda+\eta & 0 & 0 \\ -\alpha & \epsilon+\mu+\delta & 0 \\ -\eta & 0 & \mu+\nu \end{bmatrix} and$$

$$W^{-1} = \begin{bmatrix} \frac{1}{(\alpha+\mu+\lambda+\eta)} & 0 & 0 \\ \frac{1}{(\alpha+\mu+\lambda+\eta)}(\epsilon+\mu+\delta) & \frac{1}{\epsilon+\mu+\delta} & 0 \\ \frac{\eta}{(\mu+\nu)(\alpha+\mu+\lambda+\eta)} & 0 & \frac{1}{\mu+\nu} \end{bmatrix} \text{ therefore,}$$

$$ZW^{-1} = \begin{bmatrix} \frac{\beta(1-\tau)b}{\mu(\alpha+\mu+\lambda+\eta)} & 0 & 0 \\ \frac{\beta\alpha b}{\mu(\alpha+\mu+\lambda+\eta)}(\epsilon+\mu+\delta) & 0 & 0 \\ \frac{\beta(1-\tau)b\eta}{\mu(\mu+\nu)(\alpha+\mu+\lambda+\eta)} & 0 & 0 \end{bmatrix}.$$

The reproduction number is given by the spectral radius of ZW^{-1} that is

$$R_0 = \frac{\beta(1-\tau)b}{\mu(\alpha+\mu+\lambda+\eta)}.$$

STABILITY ANALYSIS FOR MUMPS

Disease-free steady state

The case where no individuals are infected to

mumps virus, disease-free steady state exist that is, $I = 0$ then $H = 0$, $Q = 0$ and $R = 0$, therefore from

$$\frac{dS}{dt} = b - \beta(1-\tau)S(t)I(t) - \mu S(t) = 0,$$
$$\Rightarrow b - \mu S^0 = 0 \Rightarrow S^0 = \frac{b}{\mu}.$$

$$J(S,I,H,Q,R) = \begin{bmatrix} -\beta(1-\tau)I - \mu & -\beta(1-\tau)S & 0 & 0 & 0 \\ \beta(1-\tau)I & \beta(1-\tau)S - \eta - \lambda - \mu - \alpha & 0 & 0 & 0 \\ 0 & \alpha & -(\delta+\epsilon+\mu) & 0 & 0 \\ 0 & \eta & 0 & -(\mu+\nu) & 0 \\ 0 & \lambda & \epsilon & \nu & -\mu \end{bmatrix}$$

Theorm 1. If $R_0 < 1$, then the disease-free steady state, E_0 is locally asymptotically stable and unstable when $R_0 > 1$.

Proof. The local stability of the steady states can be determined from the Jacobian matrix. This implies that the Jacobian matrix for the disease-free steady state is given by

$$J(E_0) = \begin{bmatrix} -\mu & \frac{-b\beta(1-\tau)}{\mu} & 0 & 0 & 0 \\ 0 & \frac{b\beta(1-\tau)}{\mu} - \eta - \lambda - \mu - \alpha & 0 & 0 & 0 \\ 0 & \alpha & -(\delta+\epsilon+\mu) & 0 & 0 \\ 0 & \eta & 0 & -(\mu+\nu) & 0 \\ 0 & \lambda & \epsilon & \nu & -\mu \end{bmatrix}$$

performing some row operation to find the eigenvalues give

$$R_1 = \frac{b\beta(1-\tau)}{b\beta(1-\tau) - \eta\mu - \lambda\mu - \mu^2 - \alpha\mu} R_2 + R_1$$

therefore,

$$J(E_0) = \begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & \frac{b\beta(1-\tau)}{\mu} - \eta - \lambda - \mu - \alpha & 0 & 0 & 0 \\ 0 & \alpha & -(\delta+\epsilon+\mu) & 0 & 0 \\ 0 & \eta & 0 & -(\mu+\nu) & 0 \\ 0 & \lambda & \epsilon & \nu & -\mu \end{bmatrix}$$

and the eigenvalues are it diagonal elements since our matrix is triangular.

$$\lambda_1, \lambda_2 = -\mu < 0,$$
$$\lambda_3 = -(\delta+\epsilon+\mu) < 0,$$
$$\lambda_4 = -(\mu+\nu) < 0,$$
$$\lambda_5 = \frac{b\beta(1-\tau)}{\mu} - \eta - \lambda - \mu - \alpha$$
$$= \frac{b\beta(1-\tau)}{\mu} - (\eta + \lambda + \mu + \alpha)$$
$$= \frac{b\beta(1-\tau)}{\mu(\eta+\lambda+\mu+\alpha)}(\eta + \lambda + \mu + \alpha) - (\eta + \lambda + \mu + \alpha)$$
$$= R_0(\eta + \lambda + \mu + \alpha) - (\eta + \lambda + \mu + \alpha)$$
$$= (\eta + \lambda + \mu + \alpha)(R_0 - 1) < 0 \text{ if } R_0 < 1.$$

Thus the system (1) has a disease-free steady state $E_0 = \left(\frac{b}{\mu}, 0,0,0,0\right)$.

To exterminate the local stability of the disease-free steady state $E_0 = \left(\frac{b}{\mu}, 0,0,0,0\right)$ by taking the Jacobian matrix in system (1) and get

Since all eigenvalues are negative when $R_0 < 1$. Hence, the disease-free steady state, E_0 is locally asymptotically stable if $R_0 < 1$ and unstable when $R_0 > 1$.

Mumps-Endemic steady state

Case 1: When $\tau_1 = 0$ then $\tau_2 = 0$ therefor, $\tau = 0$.

Aside of the disease-free steady state, the system (1) has an endemic steady state, E_* when individuals are infected to mumps virus. The endemic steady state is a positive fixed point solution where the disease still exist in the population. Defining $E_* = (S^*, I^*, H^*, Q^*, R^*)$ as the endemic steady state of the system (1) give the followings:

$$S^* = \frac{a_1}{\beta(1-\tau)}, \qquad I^* = \frac{a_2}{\beta(1-\tau)a_1},$$
$$H^* = \frac{\alpha a_2}{\beta(1-\tau)a_1 a_3}, \qquad Q^* = \frac{\eta a_2}{\beta(1-\tau)(\mu+\nu)a_1},$$
$$R^* = \frac{1}{\beta(1-\tau)\mu(\mu+\nu)a_1 a_3}[a_2(\lambda\mu^2 + \alpha\epsilon\mu + \alpha\epsilon\nu + \delta\eta\nu + \epsilon\eta\nu + \delta\lambda\mu + \delta\lambda\nu + \epsilon\lambda\mu + \epsilon\lambda\nu + \eta\mu\nu + \lambda\mu\nu)],$$

where:

$$a_1 = \alpha + \eta + \lambda + \mu,$$
$$a_2 = b\beta(1-\tau) - \alpha\mu - \eta\mu - \lambda\mu - \mu^2,$$
$$a_3 = \delta + \epsilon + \mu.$$

Existence of Endemic Steady States

In this section, we will show that when $R_0 > 1$ the endemic steady state is always exist [17, 18].

Lemma 1. If $R_0 > 1$ then the endemic steady state E_* is exist

Proof. The endemic steady state $E_* = (S^*, I^*, H^*, Q^*, R^*)$ which have been found in

previous section

$S^* = \frac{a_1}{\beta(1-\tau)}$, $I^* = \frac{a_2}{\beta(1-\tau)a_1} = \frac{b\beta(1-\tau)-\mu a_1}{\beta(1-\tau)a_1} = \frac{b-\mu S^*}{\beta(1-\tau)S^*}$.

Now, we have to prove that $b - \mu S^* > 0$ when $R_0 > 1$

$$\begin{aligned} b - \mu S^* &= b - \mu(\tfrac{a_1}{\beta(1-\tau)}) \\ &= \tfrac{\beta(1-\tau)b}{\beta(1-\tau)} - \mu(\tfrac{a_1}{\beta(1-\tau)}) \\ &= \tfrac{\frac{\beta(1-\tau)b(\mu a_1)}{\mu a_1}}{\beta(1-\tau)} - \tfrac{\mu a_1}{\beta(1-\tau)} \\ &= \tfrac{\mu a_1}{\beta(1-\tau)}R_0 - \tfrac{\mu a_1}{\beta(1-\tau)}, \text{Since } R_0 = \tfrac{\beta(1-\tau)b}{\mu(\alpha+\mu+\lambda+\eta)} = \tfrac{\beta(1-\tau)b}{\mu a_1} \\ &= \tfrac{\mu a_1}{\beta(1-\tau)}(R_0 - 1). \end{aligned}$$

Therefore, when $R_0 > 1$ it is proven that $S^* > 0$ which result $I^* > 0, H^* > 0, Q^* > 0 \; and \; R^* > 0$, so that the endemic steady states exist.

Theorm 2. If $R_0 > 1$, then the disease-endemic steady state, E_* is locally asymptotically stable and does not exist when $R_0 < 1$.

Proof. The Jacobian matrix for the endemic equilibrium is given by

$$J(E_*) = \begin{bmatrix} \frac{-b\beta(1-\tau)}{a_1} & -a_1 & 0 & 0 & 0 \\ \frac{a_2}{a_1} & 0 & 0 & 0 & 0 \\ 0 & \alpha & -a_3 & 0 & 0 \\ 0 & \eta & 0 & -(\mu+\nu) & 0 \\ 0 & \lambda & \epsilon & \nu & -\mu \end{bmatrix}.$$

Therefore, whose eigenvalues are the followings.

$\lambda_1 = -a_3 < 0,$
$\lambda_2 = -\mu < 0,$
$\lambda_3 = -\mu - \nu < 0,$
$\lambda_4 = -(\sqrt{b^2(\beta(1-\tau))^2 - 4a_2a_1^2} + b\beta(1-\tau))/2a_1,$
$\lambda_6 = (\sqrt{b^2(\beta(1-\tau))^2 - 4a_2a_1^2} - b\beta(1-\tau))/2a_1.$

Since, $\sqrt{b^2(\beta(1-\tau))^2} = b\beta(1-\tau) \rightarrow \sqrt{b^2(\beta(1-\tau))^2 - 4a_2a_1^2} < b\beta(1-\tau)$. Therefore, λ_5 and λ_6 are also negative.

The endemic state, E_* is always stable whenever it exists that is when $R_0 > 1$.

Sensitivity Analysis

The objective of sensitivity analysis is to qualitatively determine which parameters exert the greatest influence on a model's output. A parameter is considered sensitive if small perturbations in its value lead to substantial variations in the solutions of the corresponding differential equations. To conduct sensitivity analysis on a dynamical system, we assume the system consists of m compartments, denoted c_i for $i = 1,2,\dots,m$, and is governed by n parameters k_j for $j = 1,2,\dots,n$.

Representing the model balanced equations as a system of differential equations as follows (19-21):

$$\frac{dc_i}{dt} = f_i(c,k)$$

where $c \in \mathbb{R}^m$ and $k \in \mathbb{R}^n$. Non-normalization, half-normalization and full-normalization are techniques to calculate sensitivity analysis of the model.

Non-normalization is given by:

$$S_{ij} = \frac{\partial c_i(t)}{\partial k_j}.$$

Half-normalization is given by:

$$S_{ij} = \left(\frac{1}{c_i(t)}\right)\left(\frac{\partial c_i(t)}{\partial k_j}\right).$$

Full-normalization is given by:

$$S_{ij} = \left(\frac{k_j}{c_i(t)}\right)\left(\frac{\partial c_i(t)}{\partial k_j}\right).$$

Where S_{ij} is the time-dependent sensitivities of c_i with respect to each parameter value k_j.

The following are the parameter values used in the sensitivity analysis and simulation.

Table 2: Parameters value of the model.

Parameter	value	Source
b	0.5	(13)
β	0.433	(13)
α	0.005	assumed
μ	0.1	(13)
δ	0.3	assumed
ϵ	0.1	Assumed
λ	0.117	(13)
η	0.2	assumed
ν	0.371	(2)

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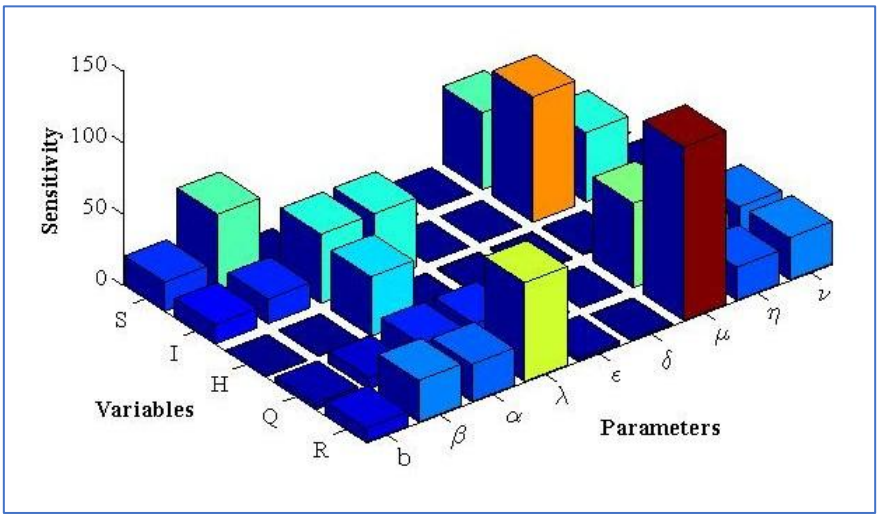

Fig. 2: Local sensitivity analysis with non-normalization technique of all variables in computational simulations using MATLAB.

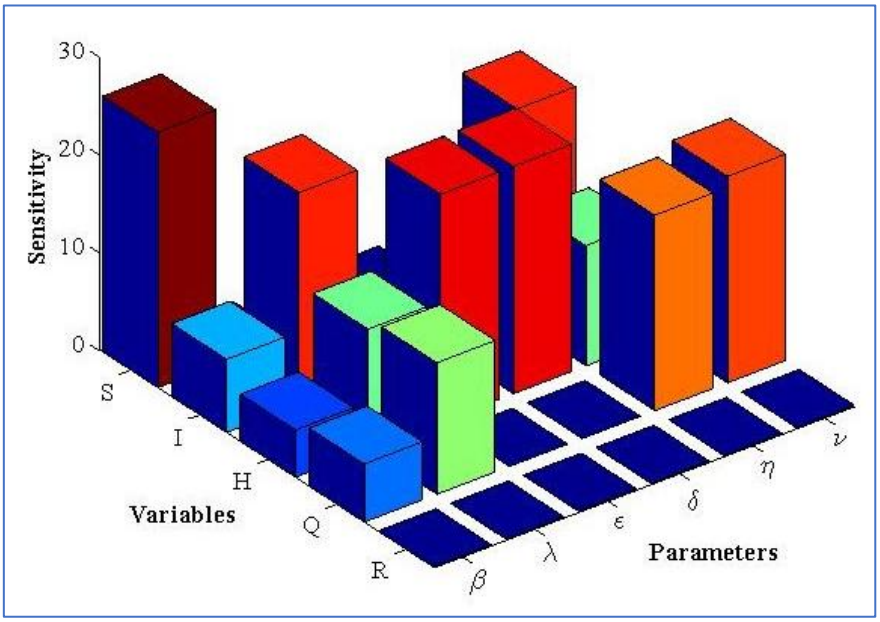

Fig. 3: Local sensitivity analysis with half-normalization technique of all variables in computational simulations using MATLAB with respect to all parameters except of b, α and μ.

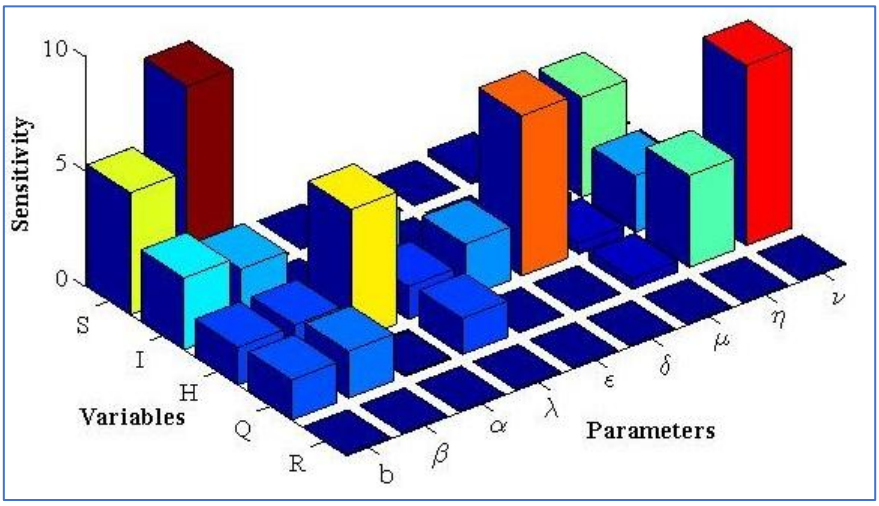

Fig. 4: Local sensitivity analysis with full-normalization technique of all variables in computational simulations using MATLAB with respect to all parameters.

In general, the simulation results indicate that several key model compartments exhibit notable sensitivity to critical parameters. For instance, the susceptible population shows sensitivity to parameters β and b, while demonstrating lower sensitivity to δ and ϵ (see Fig. 2). Additionally, infected individuals are sensitive to parameters λ, η, β, and α, whereas infected individuals experiencing hearing loss respond primarily to parameters α and δ (refer to Fig. 2 and 4). In contrast, Fig. 3 reveals that quarantined individuals are most sensitive to parameters η and ν, with reduced sensitivity to δ, while recovered individuals respond to parameters λ and μ, showing less sensitivity to δ and ϵ (see Fig. 2).

These results highlight certain contrasts and similarities in the effects of specific parameters across model states, analyzed using three distinct sensitivity analysis techniques. Based on these findings, the non-normalized sensitivity analysis technique emerges as more effective in identifying critical parameters for this model relative to alternative methods (see Fig. 2).

NUMERICAL STABILITY ANALYSIS AND SIMULATIONS

In this section, MATLAB applications are employed to examine the effects of various parameters on the stability regions and dynamics of the system (1) for both the disease-free and endemic steady states. This analysis provides a deeper understanding of the results obtained in the previous section. To illustrate the stability region of our model in Fig. 5, we select the following parameter values: $b = 0.5$, $\delta = 0.3$, $\alpha = 0.005$, $\mu = 0.1$, $\nu = 0.371$, $\tau_1 = 0$, $\tau_2 = 0$ and $\epsilon = 0.1$. This setup enables analysis of stability with respect to variations in the exposure rate β and the recovery rate λ. The results in Fig. 5 (a) and (b) demonstrate that increasing the quarantine rate η from 0.2 to 0.9 expands the stability region, highlighting η as an influential parameter in the model. Furthermore, Fig. 5 indicates that an increase in λ and a decrease in β similarly expand the stability region around the endemic steady state, while the reverse conditions produce a contraction of this region.

We choose the following parameter values to illustrate the stability region of our model: $b = 0.5$, $\delta = 0.3$, $\alpha = 0.005$, $\mu = 0.1$, $\nu = 0.371$, $\eta = 0.2$ and $\epsilon = 0.1$. In Fig. 5 (c) $\tau_1 > 0$ and $\tau_2 = 0$, whereas in Fig. 5 (d) both $\tau_1 > 0$ and $\tau_2 > 0$. The results demonstrate how vaccination expands the stability region compared to Fig. 5, highlighting the effectiveness of one and two doses of the mumps vaccine in controlling the mumps virus.

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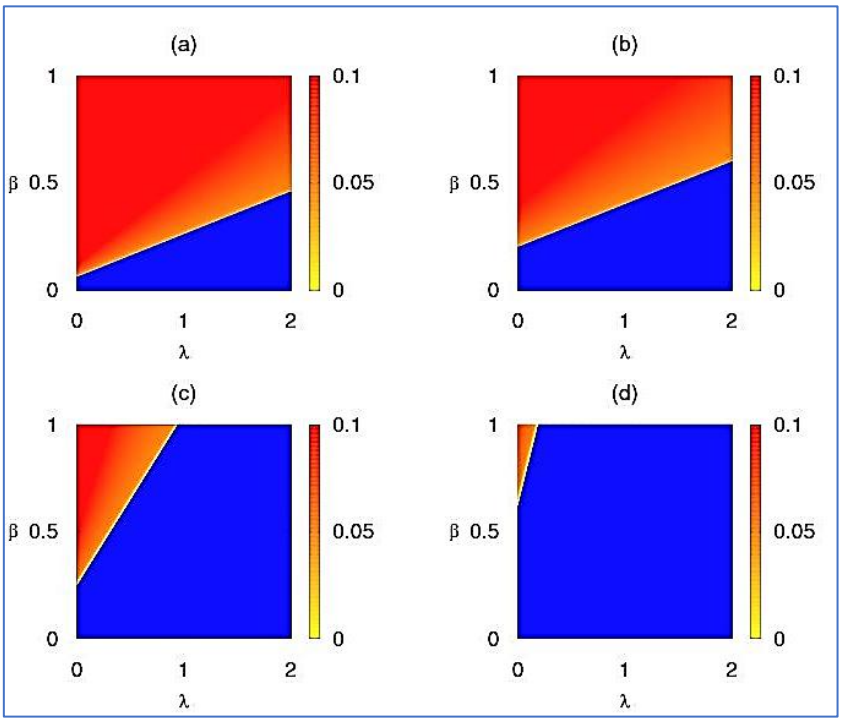

Fig. 5: Stability region of mumps virus: The portions below the curves represent the stable regions, while the portion above the curves represent the unstable regions of the model (a) $\eta = 0.2$, (b) $\eta = 0.9$, (c) with first dose of vaccine and (d) with second dose of vaccine.

In Fig.6, the model predicts that mumps will ultimately be eradicated from the population. This outcome indicates that when $R_0 < 1$, conditions are unfavorable for sustained mumps transmission. Specifically, if the birth rate among susceptible individuals is lower than the combined rates of recovery, mortality, and quarantine, the prevalence of mumps will gradually decline over time. Furthermore, the figure illustrates an increase in the recovered population as the numbers of infected and quarantined individuals decrease, suggesting that reducing the infected population and implementing timely treatment for quarantined individuals are effective measures for expediting recovery.

Conversely, if $R_0 > 1$, mumps will become endemic, and infected individuals will persist in the population. This implies that if the birth rate is higher than the natural death rate and the contact rate between mumps-infected and susceptible individuals exceeds the combined rates of recovery, death, and quarantine, the mumps outbreak will continue within the population. In addition, mumps can result in sudden, permanent, and usually unilateral (one-sided) hearing loss. This condition is thought to be caused by inflammation and damage to the cochlear structures from the mumps virus. The incidence of hearing loss from mumps is relatively low see Fig.6 and Fig.7.

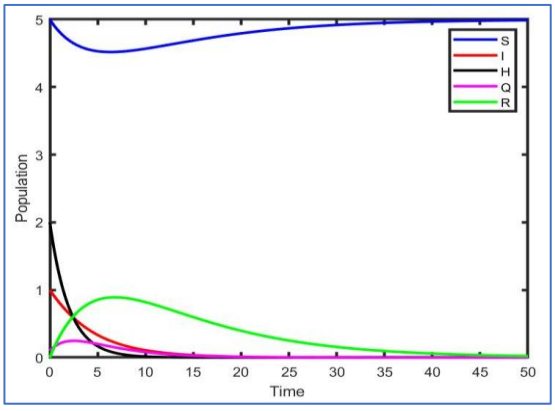

Fig.6: The dynamics of the model equation for $R_0 < 1$, mumps-free steady state.

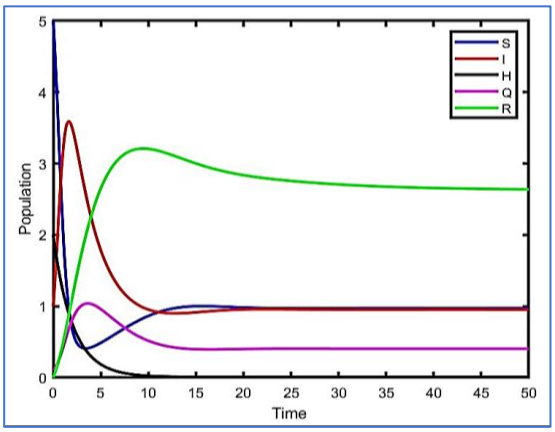

Fig.7: The dynamics of the model equation for $R_0 > 1$, endemic steady state.

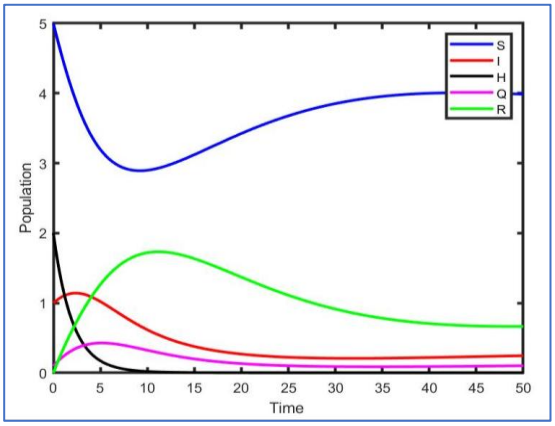

Fig. 8: The dynamics of the model equation for mumps steady state with first dose vaccination.

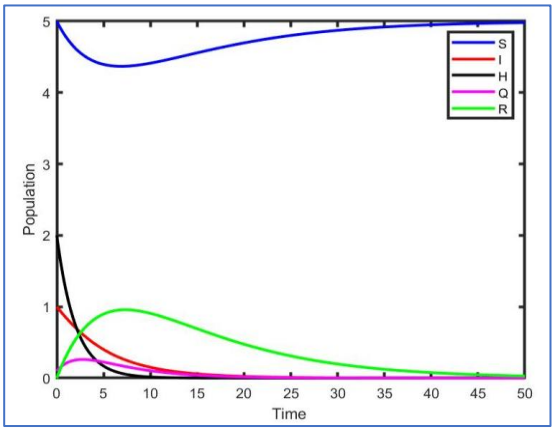

Fig. 9: The dynamics of the model equation for mumps steady state with second dose vaccination.

Fig. 8 and Fig. 9 highlight the effectiveness of the two-dose vaccination strategy in controlling mumps. While a single dose helps reduce infection rates see Fig. 8 , a second dose is much more effective in bringing mumps cases close to zero, achieving near-complete immunity within the population Fig. 9.

On the other hand, Fig. 10 if the quarantine rate is increased, the value of R_0 will decrease, and the quarantined population will approach the infected population. This indicates that quarantine can greatly decrease the transmission of mumps by separating infected individuals from those who are susceptible, thus reducing potential contact and spread. This approach can shorten the length of outbreaks and guide the population toward a mumps-free or low-level endemic state, particularly when used alongside vaccination and other preventative measures.

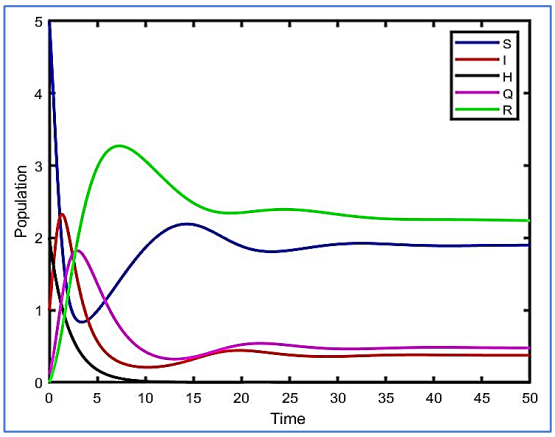

Fig. 10: The dynamics of the model equation for mumps- endemic steady state when $\eta = 0.7$.

Overall, our numerical simulations provide a comprehensive assessment of different epidemiological scenarios for mumps transmission. When $R_0 < 1$, the mumps-free equilibrium is stable, and the disease eventually disappears, as seen in Fig 6. In contrast, when $R_0 > 1$, the endemic equilibrium persists, leading to a continuous presence of mumps within the population, as shown in Fig. 7. The effectiveness of vaccination is evident, as one dose significantly reduces infection rates (Fig. 8), while two doses lead to near-elimination of the disease (Fig.9). Additionally, quarantine plays a crucial role in controlling the spread, with increased quarantine rates effectively reducing transmission and leading to a more stable, low-infection state (Fig. 10). These results

collectively highlight the importance of vaccination, quarantine, and reducing exposure rates in managing and ultimately eradicating mumps.

DISCUSSION

This study presents a mathematical model of mumps virus and its impact on hearing loss using ordinary differential equations. The SIHQR model divides the population into five compartments: susceptible individuals (S), infected individuals (I), individuals experiencing hearing loss (H), quarantined individuals (Q), and recovered individuals (R). Analyzing the model established the existence and stability of the mumps steady states. Specifically, the mumps-free state was shown to be locally asymptotically stable when the basic reproductive number is less than one, while the endemic state remains stable when the basic reproductive number is greater than one. This confirms that the model is well-posed and accurately represents the disease dynamics.

Furthermore, the study assesses the role of delay treatment in the spread of the disease, a crucial aspect often overlooked in previous models. The findings emphasize the significance of vaccination and quarantine strategies in controlling mumps transmission and mitigating complications such as hearing loss. The sensitivity analysis highlights the impact of critical parameters on the disease progression (see Fig. 2– 4), reinforcing the necessity of timely interventions.

From a broader perspective, these results have important public health implications. Understanding the stability conditions of mumps steady states can help policymakers implement targeted interventions to control outbreaks. The model can also serve as a foundation for further studies on optimizing vaccination coverage and quarantine measures to prevent long-term complications. Ultimately, this research contributes to better disease management strategies, improving both prevention and treatment outcomes.

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Author contribution: Authors contributed equally in the study.

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