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# Solving Fredholm Integro-Differential Equations of Mixed-Order by $\alpha$ -Dickson Matrix-Collocation Method

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## ABSTRACT

In this article, we utilize  $(\alpha \in \mathbb{R}^+)$ -Dickson polynomials to improve a method for spectral collocation that is used to numerically solve mixed classical and fractional-order integro-differential equations of the Fredholm type with variable coefficients under boundary conditions (IFDEs-CF) in the Caputo sense. The approximate solutions are assumed in the form of the truncated Dickson polynomial series. This innovative method transforms the equation with conditions into a linear algebraic system for the unknown Dickson coefficients using a matrix technique. Solving this system determines the coefficients of the assumed solution. We ultimately analyze the results with respect to the  $\alpha$  parameter in the Dickson polynomials. The behavior of the solutions is monitored in terms of the  $\alpha$ -Dickson coefficients depending on different inputs of  $\alpha$ . This approach is computationally attractive, and we provide illustrative examples with usage explanations. Additionally, we provide specific examples to demonstrate the method's accuracy, and we use least-squares error minimization to reduce error terms within the given domain. Finally, we develop the most common application of numerical approaches in a Python program.

**Keywords:** Fredholm integral equations, fractional derivative, Dickson polynomial, Caputo derivative, matrix technique, Standard collocation points.

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# حل معادلات فريدهولم التكاملية التفاضلية من الدرجة المختلطة باستخدام طريقة تجميع مصفوفات

ألفا-ديكسون

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## الملخص

في هذه المقالة، نستخدم كثيرات حدود  $(+\alpha \in \mathbb{R})$ -ديكسون لتحسين طريقة التجميع الطيفي المستخدمة لحل المعادلات التفاضلية المختلطة الكلاسيكية والكسرية من نوع فريدهولم عددياً مع معاملات متغيرة في ظل ظروف حدودية (IFDES-CF) بمعنى كابوتو. تُفترض الحلول التقريبية في شكل سلسلة كثيرات حدود ديكسون المبتورة. تعتمد هذه الطريقة المبتكرة على تحويل المعادلة المعطاة مع الشروط إلى نظام معادلات خطي جبري مع معاملات ديكسون غير معروفة باستخدام تقنية المصفوفة. يحدد حل هذا النظام معاملات الحل المقترض. يتم تحليل النتائج في النهاية فيما يتعلق بمعامل  $\alpha$  في كثيرات حدود ديكسون. تتم مراقبة سلوك الحلول من حيث معاملات  $\alpha$ -ديكسون اعتماداً على مدخلات مختلفة لـ  $\alpha$ . هذا النهج جذاب من الناحية الحسابية، ويتم توفير أمثلة توضيحية مع شرح الاستخدام. بالإضافة إلى ذلك، نقدم أمثلة محددة لتوضيح دقة الطريقة، ونستخدم منهجية خطأ المربعات الصغرى لتقليل حدود الخطأ ضمن النطاق المحدد. وأخيراً، تم تطوير التطبيق الأكثر شيوعاً المقترح للمناهج العددية باستخدام لغة بايثون.

الكلمات المفتاحية / معادلات فريدهولم التكاملية، المشتقة الكسرية، متعددة حدود ديكسون، مشتقة كابوتو، تقنية المصفوفة، نقاط التجميع القياسية.

## 1- INTRODUCTION

Differential and integral equations make up integro-differential equations (IDEs). These equations are fundamental to applied mathematics, engineering, mechanics, physics, chemistry, potential theory, dynamics, and ecology. Because these equations are typically challenging to solve analytically, they usually require numerical approaches (1-3). Researchers across a wide range of scientific and technical domains have recently shown strong interest in fractional differential equations and integro-fractional differential equations (IFDEs). This interest stems from their ability to effectively model physical phenomena that depend not only on the current state but also on their historical behavior. While accurate modeling is essential, the methods used to solve these equations and the reliability of these solutions are equally important, particularly in identifying critical points such as rapid divergence,

convergence, or bifurcation. Consequently, high-precision solutions are consistently required. To address this need, several methods have been developed to solve fractional-order differential and integro-differential equations. Among the most widely utilized techniques are the Adomian decomposition method (4), variational iteration method (5), fractional differential transform method (6), fractional difference method (7), and power series method (8). Various studies (9-13) have applied the Bessel polynomial, Laguerre, Legendre, Fourier, Hermite methods and spline functions to linear integral and differential and integro-fractional difference equations and associated systems, respectively. Additionally, we provide integral and fractional derivatives, as well as a fractional differential system with initial conditions involving fractional terms, in (14, 15). Furthermore, we used Dickson First-kind polynomials to approximate integro-differential equations of fractional order,

LFIDEs, differential equations, and fractional differential equations (16-22). In this work, we develop a new method for approximating solutions to Fredholm-type mixed-order (classical and fractional) integro-differential equations by employing collocation methods and extending the first-kind  $\alpha$ -Dickson polynomials in matrix form.

In this paper, the core concepts from previous works are extended through a new methodological approach that relies on standard collocation points and first-kind  $\alpha$ -Dickson polynomials in matrix form and is applied to problems consisting of linear Fredholm Integro-Differential Equations of multi-high Classical and Fractional orders (FIDEs-CF) with variable coefficients:

$$p(x)y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) {}^C D_x^{\sigma_i} y(x) + p_n(x)y(x) \\ = f(x) + \sum_{j=0}^m \lambda_j \int_a^b \mathcal{K}_j(x,s) {}^C D_s^{\beta_j} y(s) ds \dots (1)$$

Under the boundary conditions:

$$\sum_{\ell=0}^{\mu-1} [\mathcal{G}_{k\ell} y^{(\ell)}(a) + \mathcal{H}_{k\ell} y^{(\ell)}(b)] = \vartheta_k, \text{ for all } k \\ = 0, 1, \dots, \mu-1 \dots (2)$$

Where the functions:  $\{p_i\}_{i=0}^n, f$  and  $p \in C([a, b], \mathbb{R})$  and  $\{\mathcal{K}_j\}_{j=0}^m \in C(\mathfrak{N}, \mathbb{R}), \mathfrak{N} = \{(x, s): a \leq x, s \leq b\}$  are known; in addition,  $y(x)$  is an unknown continuous function. While given arbitrary real constants  $\vartheta_k, \mathcal{G}_{k\ell}, \mathcal{H}_{k\ell}$  for each  $k, \ell = 0, 1, \dots, \mu-1$ . Where the fractional orders:  $\sigma_{n-1} > \dots > \sigma_1 > \sigma_0 > 0$  with  $\sigma_n = 0$  and  $\beta_m > \beta_{m-1} > \dots > \beta_1 > \beta_0 = 0$ , Since  $\sigma_i < n$  with  $\beta_j \in \mathbb{R}^+$ , for all  $i = 0, 1, \dots, n-1$  and  $j = 0, 1, \dots, m$ ; and the  $\mu = \max \{n, \omega_j^{\beta_j}\}$  where  $\omega_j^{\beta_j} - 1 < \beta_j \leq \omega_j^{\beta_j}$ ,  $\omega_j^{\beta_j} = [\beta_j]$  for all  $j = 0, 1, \dots, m$ , and  $\omega_i^{\sigma_i} - 1 < \sigma_i \leq \omega_i^{\sigma_i}$ ,  $\omega_i^{\sigma_i} = [\sigma_i]$ , for all  $i = 0, 1, \dots, n-1$  and  $n \in \mathbb{Z}^+, m \in \mathbb{Z}^+ \cup \{0\}$ .

## 2- FUNDAMENTAL CONSIDERATIONS

### 2.1- Fractional Calculus:

One important area of mathematics, Fractional Calculus (FC), concerns differentiating and

integrating into any arbitrary order. In recent years, numerous definitions of FC have emerged. (23-26). This section introduces the fundamental concepts and properties of fractional calculus (FC) used later in the article.

**Definition 1.** (23) The space  $C_\delta(I)$ ,  $\delta \in \mathbb{R}$ , contains all real-valued functions  $y$  defined on a closed bounded interval  $[a, b] = I$ , if there exists a real number  $\omega > \delta$  such that  $y(x) = (x - a)^\omega y_0(x)$  where  $y_0 \in C(I)$ . Moreover,  $y(x)$  is considered to be in the space  $C_\delta^n(I)$ , where  $n \in \mathbb{Z}^+ \cup \{0\}$ , if and only if the  $n$ -th derivative of  $y$  with respect to  $x$ ,  $y^{(n)}$ , belongs to  $C_\delta(I)$ .

**Definition 2.** (27) The  $\alpha$ -Riemann-Liouville fractional integral on  $[a, b]$  for a function  $y \in C_\delta, \delta \geq -1$ , represented as the operator  ${}_a J_x^\alpha$ , of order  $\alpha > 0$ , is defined as:

$${}_a J_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t) dt, \alpha \in \mathbb{R}^+$$

Here,  $\Gamma(\cdot)$  denotes the gamma function and the operator  ${}_a J_x^\alpha$  becomes the identity when the zeroth fractional order, that is  ${}_a J_x^0 y(x) = y(x)$ . Also, it has a commutativity property, that is, for any arbitrary fractional orders  $(\alpha$  and  $\beta) \in \mathbb{R}^+$ ,  ${}_a J_x^\alpha {}_a J_x^\beta y(x) = {}_a J_x^\beta {}_a J_x^\alpha y(x) = {}_a J_x^{\alpha+\beta} y(x)$ . And  ${}_a J_x^\alpha$  is a linear operator.

**Definition 3.** (28) For a function on the interval  $[a, b]$ ,  $y(x) \in C_{-1}^\omega$ , and  $\omega = [\alpha], \alpha \geq 0$ , the Riemann-Liouville fractional derivative operator,  ${}_a^R D_x^\alpha$ , is often stated as:

$${}_a^R D_x^\alpha y(x) = D_x^\omega {}_a J_x^{\omega-\alpha} y(x) \\ = \begin{cases} \frac{\partial^\omega}{\partial x^\omega} \left( \int_a^x \frac{(x-t)^{\omega-\alpha-1}}{\Gamma(\omega-\alpha)} y(t) dt \right), & \alpha \in \mathbb{R}^+ \\ \frac{d^\alpha}{dx^\alpha} y(x), & \alpha = [\alpha] \in \mathbb{Z}^+ \cup \{0\} \end{cases}$$

where  $\omega - 1 < \alpha \leq \omega (= [\alpha] \in \mathbb{Z}^+)$ , and  ${}_a^R D_x^\alpha$  are linear operators.

**Definition 4** (29). For a function on the interval  $[a, b]$ ,  $y(x) \in C_{-1}^\omega$ , and  $\omega = [\alpha], \alpha \geq 0$ , the Caputo fractional derivative operator,  ${}_a^C D_x^\rho$ , is often stated as:

$${}_a^C D_x^\alpha y(x) = {}_a^J x^{\omega-\alpha} D_x^\omega y(x) \\ = \begin{cases} \int_a^x \frac{(x-t)^{\omega-\alpha-1}}{\Gamma(\omega-\alpha)} \frac{d^\omega y(t)}{dt^\omega} dt, & \alpha \in \mathbb{R}^+ \\ \frac{d^\alpha}{dx^\alpha} y(x), & \alpha = [\alpha] \in \mathbb{Z}^+ \cup \{0\} \end{cases}$$

where  $\omega - 1 < \alpha \leq \omega (= [\alpha] \in \mathbb{Z}^+)$ . The fundamental properties of the Caputo fractional derivative are: The Caputo-type fractional derivative is a linear operator for any fractional order, and the  $\alpha$ -Caputo fractional derivative of a constant ( $A$ ) is equal to zero, i.e.  ${}_a^C D_x^\alpha A = 0$ .

**Lemma 1** <sup>(25)</sup>. If we have  $x > a$  and for  $y(x) = (x-a)^\nu$  where  $\nu \neq -\omega$  with  $\omega = [\alpha]$ , then the following statement holds:

$${}_a^J x^\alpha y(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\rho+1)} (x-a)^{\nu+\alpha}, \alpha \in \mathbb{R}.$$

**Lemma 2** <sup>(24)</sup>. The function  $y(x) = (x-a)^\nu$ , for  $\nu \geq 0$ , has a Caputo derivative of order  $\alpha \geq 0$  with

$$\mathcal{D}_r(x, \alpha) = \begin{cases} \sum_{\rho=0}^{[r/2]} \frac{r}{r-\rho} \binom{r-\rho}{\rho} (-\alpha)^\rho x^{(r-2\rho)}, & r \geq 1, -\infty < x < \infty \\ 2, & r = 0 \end{cases}$$

here  $[r/2]$  is the greatest integer  $\leq r/2$ , denotes  $\mathcal{D}_r(x, 0)$  become power polynomials for  $\alpha = 0$ , and they are described by the monic polynomial  $x^r$  <sup>(30)</sup>. The recurrence relation of  $\alpha$ -Dickson polynomials  $\mathcal{D}_r(x, \alpha)$  for all  $r = 0, 1, 2, \dots$  is formulated as:

$\mathcal{D}_r(x, \alpha) = x \mathcal{D}_{r-1}(x, \alpha) - \alpha \mathcal{D}_{r-2}(x, \alpha)$ ,  $r \geq 2$  with two initial polynomials:  $\mathcal{D}_0(x, \alpha) = 2$  and  $\mathcal{D}_1(x, \alpha) = x$ . Subclasses of the  $\alpha$ -a number of combinatorial polynomial families form Dickson polynomials, and their dilates, such as the Fermat-

$$\mathcal{D}(x, \alpha) = [\mathcal{D}_0(x, \alpha) \quad \mathcal{D}_1(x, \alpha) \quad \dots \quad \mathcal{D}_N(x, \alpha)] \quad \text{and} \quad X(x) = [1 \quad x \quad \dots \quad x^N]$$

For odd values of  $N$ :

$$\mathcal{C}^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1} \binom{1}{0} (-\alpha)^0 & 0 & 0 & \dots & 0 \\ \frac{2}{1} \binom{1}{1} (-\alpha)^1 & 0 & \frac{2}{2} \binom{2}{0} (-\alpha)^0 & 0 & \dots & 0 \\ 0 & \frac{3}{2} \binom{2}{1} (-\alpha)^1 & 0 & \frac{3}{3} \binom{3}{0} (-\alpha)^0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{N}{[N/2]} \binom{[N/2]}{[N/2]} (-\alpha)^{[N/2]} & 0 & \frac{N}{[N/2]+1} \binom{[N/2]+1}{[N/2]-1} (-\alpha)^{[N/2]-1} & \dots & \frac{N}{N} \binom{N}{0} (-\alpha)^0 \end{bmatrix}_{N+1 \times N+1}$$

For even values of  $N$ :

$\omega = [\alpha]$ , which is formed as: For  $\nu \in \{0, 1, 2, \dots, \omega-1\}$ :  ${}_a^C D_x^\alpha y(x) = 0$  and for  $\nu \in \mathbb{N}$  and  $\nu \geq \omega$  or  $\nu \notin \mathbb{N}$  and  $\nu > \omega-1$ :

$${}_a^C D_x^\alpha y(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} (x-a)^{\nu-\alpha}$$

**Lemma 3** <sup>(24)</sup>. Assume that  $\alpha \geq 0$ ,  $\omega = [\alpha]$ . Furthermore, suppose that.  $y \in C_{-1}^\omega$  on any closed boundary interval  $[a, b]$ . So, the  $\alpha$ -Caputo fractional derivative of  $y$  is continuous on it and  $\lim_{x \rightarrow a} [{}_a^C D_x^\alpha y(x)] = 0$ .

## 2.2- $\alpha$ -Dickson Polynomial of the First Kind with Fundamental Matrix <sup>(16-19, 21, 22)</sup>

The first kind of the  $r$ -th degree Dickson polynomials for a real parameter  $\alpha \in \mathbb{R}$  is defined as:

$$2, r = 0$$

Lucas polynomials  $FL_r\left(\frac{x}{3}\right)$  for  $\alpha = 2$ , the Lucas polynomials  $L_r(x)$  and the Pell-Lucas polynomials  $Q_r(x/2)$ , for  $\alpha = -1$ , and the first kind Chebyshev polynomials  $2T_r(x/2)$  for  $\alpha = 1$ . Consequently, we can represent the  $\alpha$ -Dickson Polynomials of the first kind,  $\mathcal{D}_r(x, \alpha)$  for  $r \geq 1$ , in matrix form as:

$$\mathcal{D}(x, \alpha) = X(x) \mathcal{C}(\alpha) \text{ or } \mathcal{D}^T(x, \alpha) = \mathcal{C}^T(\alpha) X^T(x) \dots (3)$$

where

$$\begin{aligned}
 & \mathcal{C}^T(\alpha) \\
 &= \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1} \binom{1}{0} (-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1} \binom{1}{1} (-\alpha)^1 & 0 & \frac{2}{2} \binom{2}{0} (-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2} \binom{2}{1} (-\alpha)^1 & 0 & \frac{3}{3} \binom{3}{0} (-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{N}{\frac{N}{2}} \binom{\frac{N}{2}}{\frac{N}{2}} (-\alpha)^{\frac{N}{2}} & 0 & \frac{N}{\frac{N}{2}+1} \binom{\frac{N}{2}+1}{\frac{N}{2}} (-\alpha)^{\frac{N}{2}-1} & 0 & \cdots & \frac{N}{N} \binom{N}{0} (-\alpha)^0 \end{bmatrix}_{N+1 \times N+1} \\
 & \quad CD(x, n) + FD(x, \{\sigma_i\}_{i=0}^n) \\
 & \quad = f(x) + IF\left(x, \{\beta_j\}_{j=0}^m\right) \dots (4)
 \end{aligned}$$

### 3- FUNDAMENTAL MATRIX RELATIONS

We now return to equation (1) and rewrite it in the following form:

where the mixed-part order derivative is

$$CD(x, n) = p(x)y^{(n)}(x) \quad \text{and} \quad FD(x, \{\sigma_i\}_{i=0}^n) = \sum_{i=0}^n p_i(x) {}^C D_x^{\sigma_i} y(x) \dots (4.1)$$

and the Fredholm integral part:

$$\begin{aligned}
 & IF(x, \{\beta_j\}_{j=0}^m) \\
 &= \sum_{j=0}^m \lambda_j \int_a^b \mathcal{K}_j(x, s) {}^C D_s^{\beta_j} y(s) ds \dots (4.2)
 \end{aligned}$$

$$y(x) \cong \hat{y}_N(x) = \sum_{r=0}^N a_r \mathcal{D}_r(x, \alpha) \dots (5)$$

To get an approximate solution in form (5), the unknown  $\alpha$ -Dickson coefficients are denoted by  $a_r, r = 0, 1, 2, \dots, N$  must be determined.

#### 3.1 Matrix Relation for the Mixed Order Derivative Part $CD(x, n)$ and $FD(x, \{\sigma_i\}_{i=0}^n)$ :

We assumed that the  $N$ -truncated  $\alpha$ -Dickson series expansion of the type (5) is present in the solution of equation (1). Consequently, (5) explains the solution, and its mixed-order derivatives may be converted into matrix forms:

$$y(x) = \mathcal{D}(x, \alpha) \mathcal{A} \dots (6)$$

such that

$$\mathcal{D}(x, \alpha) = [\mathcal{D}_0(x, \alpha) \quad \mathcal{D}_1(x, \alpha) \quad \cdots \quad \mathcal{D}_N(x, \alpha)] \quad \text{and} \quad \mathcal{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$

Hence, substituting the representation of the  $\alpha$ -Dickson polynomials of the first kind as in equation (3) for each part into equation (6), we obtain:

$$y(x) = \mathbf{X}(x) \mathcal{C}(\alpha) \mathcal{A}$$

Consequently, for  $n$ -classical derivative part representation as the following expression:

$$y^{(n)}(x) = \frac{d^n}{dx^n} \mathcal{D}(x, \alpha) \mathcal{A}$$

where

$$\begin{aligned}
 & \frac{d^n}{dx^n} \mathcal{D}(x, \alpha) \\
 &= [\mathcal{D}_0^{(n)}(x, \alpha) \quad \mathcal{D}_1^{(n)}(x, \alpha) \quad \cdots \quad \mathcal{D}_N^{(n)}(x, \alpha)]
 \end{aligned}$$

Hence, we also have

$$y^{(n)}(x) = \mathbf{X}^{(n)}(x) \mathcal{C}(\alpha) \mathcal{A} \dots (7)$$

At this point, we determine the relationship between matrix  $\mathbf{X}(x)$  and its derivative.  $\mathbf{X}^{(1)}(x)$ . The following is obtained by taking the first derivative of  $\mathbf{X}(x) = [1 \quad x \quad \cdots \quad x^N]$  with respect to  $x$ :

$$\begin{aligned}
 \frac{d}{dx} \mathbf{X}(x) &= \mathbf{X}^{(1)}(x) \\
 &= [0 \quad 1 \quad 2x \quad 3x^2 \quad \cdots \quad Nx^{N-1}]
 \end{aligned}$$

In matrix form, it's possible to write consequently:

$$\mathbf{X}^{(1)}(x) = \mathbf{X}(x) \mathcal{H}^T \dots (8)$$

where

$$\mathcal{H}^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1) \times (N+1)}$$

Once more, we get the recurrence relation from equation (8):

$$\begin{aligned} \mathbf{X}^{(0)}(x) &= \mathbf{X}(x) \\ \mathbf{X}^{(1)}(x) &= \mathbf{X}(x)\mathcal{H}^T \\ \mathbf{X}^{(2)}(x) &= \mathbf{X}^{(1)}(x)\mathcal{H}^T = \mathbf{X}(x)[\mathcal{H}^T]^2 \\ &\vdots \\ \mathbf{X}^{(n)}(x) &= \mathbf{X}^{(n-1)}(x)\mathcal{H}^T = \mathbf{X}(x)[\mathcal{H}^T]^n \end{aligned}$$

Here, note that  $[\mathcal{H}^T]^0 = [\mathbf{I}]_{N+1}$ . Assume it is true for order  $m-1$  to demonstrate the final recurrence relations using the mathematical induction concept  $\mathbf{X}^{(m-1)}(x) = \mathbf{X}(x)[\mathcal{H}^T]^{m-1}$ . Thus, for order  $m$  is

$$\mathbf{X}^{(m)}(x) = \mathbf{X}^{(m-1+1)}(x) = \frac{d^{m-1}}{dx^{m-1}} \left( \frac{d}{dx} \mathbf{X}(x) \right)$$

$$\begin{aligned} &= \frac{d^{m-1}}{dx^{m-1}} (\mathbf{X}(x)\mathcal{H}^T) \\ &= \mathbf{X}^{(m-1)}(x)\mathcal{H}^T \\ &= \mathbf{X}(x)[\mathcal{H}^T]^{m-1}\mathcal{H}^T \\ &= \mathbf{X}(x)[\mathcal{H}^T]^m \quad \dots (9) \end{aligned}$$

$$\begin{aligned} {}_a J_x^{\omega_i^\sigma - \sigma_i} \mathbf{X}(x) &= {}_a J_x^{\omega_i^\sigma - \sigma_i} \{ [1 \quad x \quad \cdots \quad x^N]_{1 \times (N+1)} \} \\ &= \left[ \frac{\Gamma(1)}{\Gamma(\omega_i^\sigma - \sigma_i + 1)} x^{\omega_i^\sigma - \sigma_i + 0} \quad \frac{\Gamma(2)}{\Gamma(\omega_i^\sigma - \sigma_i + 2)} x^{\omega_i^\sigma - \sigma_i + 1} \quad \cdots \quad \frac{\Gamma(N+1)}{\Gamma(\omega_i^\sigma - \sigma_i + N + 1)} x^{\omega_i^\sigma - \sigma_i + N} \right]_{1 \times (N+1)} \\ &= x^{\omega_i^\sigma - \sigma_i} [1 \quad x \quad \cdots \quad x^N] \mathcal{M}(\omega_i^\sigma - \sigma_i) \end{aligned}$$

where

$$\mathcal{M}(\omega_i^\sigma - \sigma_i) = \text{diag} \left( \left[ \frac{\Gamma(1)}{\Gamma(\omega_i^\sigma - \sigma_i + 1)} \quad \frac{\Gamma(2)}{\Gamma(\omega_i^\sigma - \sigma_i + 2)} \quad \cdots \quad \frac{\Gamma(N+1)}{\Gamma(\omega_i^\sigma - \sigma_i + N + 1)} \right]_{N+1} \right) \quad \dots (11)$$

Thus, for all  $i = 0, 1, 2, \dots, n-1$  generally, we obtain:

$$\begin{aligned} {}_a^C D_x^{\sigma_i} y(x) &= x^{\omega_i^\sigma - \sigma_i} \mathbf{X}(x) \mathcal{M}(\omega_i^\sigma - \sigma_i) \\ &\quad - \sigma_i [\mathcal{H}^T]^{\omega_i^\sigma} \mathcal{C}(\alpha) \mathcal{A} \quad \dots (12) \\ CD(x, n) + FD(x, \{\sigma_i\}_{i=0}^n) \end{aligned}$$

$$\begin{aligned} &= \left[ p(x) \mathbf{X}(x) [\mathcal{H}^T]^n + \sum_{i=0}^{n-1} p_i(x) x^{\omega_i^\sigma - \sigma_i} \mathbf{X}(x) \mathcal{M}(\omega_i^\sigma - \sigma_i) [\mathcal{H}^T]^{\omega_i^\sigma} \right. \\ &\quad \left. + p_n(x) \mathbf{X}(x) \right] \mathcal{C}(\alpha) \mathcal{A} \quad \dots (13) \end{aligned}$$

### 3.2 Matrix Representation of the Fredholm Integral Part $IF(x, \{\beta_j\}_{j=0}^m)$ :

Now, we give the kernel function.  $\mathcal{K}_j(x, s), j = 0, 1, 2, \dots, m$  for the Fredholm integral part

After applying equation (9) for every  $n \in \mathbb{Z}^+ \cup \{0\}$ , equation (7) became:

$$y^{(n)}(x) = \mathbf{X}(x) [\mathcal{H}^T]^n \mathcal{C}(\alpha) \mathcal{A} \quad \dots (10)$$

Second, we may modify the  $(n-1)$ -fractional terms in equation (4) by using equation (7) with (9) and using the Caputo derivative definition 4 with Lemmas 1 and 2,  $FD(x, \{\sigma_i\}_{i=0}^{n-1}), \omega_i^\sigma - 1 < \sigma_i \leq \omega_i^\sigma; \omega_i^\sigma = [\sigma_i]$ , for all  $i = 0, 1, 2, \dots, n-1$  to matrix form:

$$\begin{aligned} {}_a^C D_x^{\sigma_i} y(x) &= {}_a^C D_x^{\sigma_i} \{ \mathbf{X}(x) \mathcal{C}(\alpha) \mathcal{A} \} \\ &= {}_a J_x^{\omega_i^\sigma - \sigma_i} D_x^{\omega_i^\sigma} \{ \mathbf{X}(x) \mathcal{C}(\alpha) \mathcal{A} \} \\ &= {}_a J_x^{\omega_i^\sigma - \sigma_i} \{ \mathbf{X}(x) [\mathcal{H}^T]^{\omega_i^\sigma} \mathcal{C}(\alpha) \mathcal{A} \} \end{aligned}$$

Since, by applying the R-L fractional integral and Lemma 1, we obtain:

By substituting expressions (7), (10) and (12) into the mixed-order derivative parts:  $CD(x, n)$  and  $FD(x, \{\sigma_i\}_{i=0}^n)$  of equation (4), we obtain:

$IF(x, \{\beta_j\}_{j=0}^m)$  in the N-truncated  $\alpha$ -Dickson expansion and the N-truncated Taylor expansion forms around  $(0, 0)$ , separately for each  $j = 0, 1, \dots, m$ :

$$\mathcal{K}_j(x, s) = \left. \begin{aligned} &\sum_{m=0}^N \sum_{n=0}^N {}_a\mathcal{K}_{mn}^j \mathcal{D}_m(x, \alpha) \mathcal{D}_n(s, \alpha) \\ &\text{and} \\ &\sum_{m=0}^N \sum_{n=0}^N {}_t\mathcal{K}_{mn}^j x^m s^n \end{aligned} \right\} \dots (14)$$

where

$${}_t\mathcal{K}_{mn}^j = \frac{1}{m!n!} \frac{\partial^{m+n} k_j(0,0)}{\partial x^m \partial s^n},$$

$$j = 0, 1, \dots, m, m, n = 0, 1, \dots, N$$

we can write the matrix forms of  $\mathcal{K}_j(x, s)$  in equation (14) as follows, respectively, for all  $j = 0, 1, \dots, m; m, n = 0, 1, \dots, N$ :

$$\begin{aligned} \mathcal{K}_j(x, s) &= \mathcal{D}(x, \alpha) \mathcal{K}_d^j \mathcal{D}^T(s, \alpha), \mathcal{K}_d^j \\ &= [{}_a\mathcal{K}_{mn}^j] \dots (15) \end{aligned}$$

and

$$\mathcal{K}_j(x, s) = \mathbf{X}(x) \mathcal{K}_t^j \mathbf{X}^T(s), \mathcal{K}_t^j = [{}_t\mathcal{K}_{mn}^j] \dots (16)$$

The correlation between the  $\alpha$ -Dickson and Taylor coefficients of the kernel function  $\mathcal{K}_j(x, s), j = 0, 1, \dots, m$  is obtained by applying relation (3) and the equality of relations (15, 16):

$$\begin{aligned} \mathbf{X}(x) \mathcal{K}_t^j \mathbf{X}^T(s) &= \mathcal{D}(x, \alpha) \mathcal{K}_d^j \mathcal{D}^T(s, \alpha) \\ &= \mathbf{X}(x) \underbrace{\mathcal{C}(\alpha) \mathcal{K}_d^j \mathcal{C}^T(\alpha)}_{\mathcal{K}_t^j} \mathbf{X}^T(s) \end{aligned}$$

$$\begin{aligned} IF(x, \{\beta_j\}_{j=0}^m) &= \sum_{j=0}^m \lambda_j \int_a^b \mathcal{K}_j(x, s) {}_a^c D_s^{\beta_j} y(s) ds \\ &= \sum_{j=0}^m \lambda_j \int_a^b \mathcal{D}(x, \alpha) \mathcal{K}_d^j \mathcal{D}^T(s, \alpha) s^{\omega_j^\beta - \beta_j} \mathbf{X}(s) \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \mathcal{C}(\alpha) \mathcal{A} ds \\ &= \sum_{j=0}^m \lambda_j \int_a^b \mathbf{X}(x) \mathcal{C}(\alpha) \mathcal{K}_d^j \mathcal{C}^T(\alpha) \mathbf{X}^T(s) s^{\omega_j^\beta - \beta_j} \mathbf{X}(s) \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \mathcal{C}(\alpha) \mathcal{A} ds \\ &= \sum_{j=0}^m \lambda_j \mathbf{X}(x) \mathcal{C}(\alpha) \mathcal{K}_d^j \mathcal{C}^T(\alpha) \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \mathcal{C}(\alpha) \mathcal{A} \dots (19) \end{aligned}$$

where, for each  $j = 0, 1, 2, \dots, m$ :

$$\mathcal{Q}_j^\beta = \int_a^b \mathbf{X}^T(s) s^{\omega_j^\beta - \beta_j} \mathbf{X}(s) ds$$

while, for all  $j = 0, 1, 2, \dots, m$ , and  $\kappa, \rho = 0, 1, 2, \dots, N$ :

$$\begin{aligned} \mathcal{Q}_j^\beta &= [q_{\kappa\rho}^{\beta,j}], q_{\kappa\rho}^{\beta,j} \\ &= \frac{b^{\omega_j^\beta - \beta_j + \kappa + \rho + 1} - a^{\omega_j^\beta - \beta_j + \kappa + \rho + 1}}{\omega_j^\beta - \beta_j + \kappa + \rho + 1} \end{aligned}$$

in (19) for the matrix relation (17), the matrix form (20) may be obtained:

So, for each  $j = 0, 1, 2, \dots, m$

$$\begin{aligned} \mathcal{K}_t^j &= \mathcal{C}(\alpha) \mathcal{K}_d^j \mathcal{C}^T(\alpha) \\ &\text{that is} \dots (17) \\ \mathcal{K}_d^j &= [\mathcal{C}(\alpha)]^{-1} \mathcal{K}_t^j [\mathcal{C}^T(\alpha)]^{-1} \end{aligned}$$

However, using equation (12), transform.  ${}_a^c D_s^{\beta_j} y(s)$  in the same way, where  $\omega_j^\beta - 1 < \beta_j \leq \omega_j^\beta$ , ( $\omega_j^\beta = [\beta_j]$ ) for all  $j = 0, 1, 2, \dots, m$ . The following results from utilizing Lemma 1 and the Caputo definition for fractional orders  $\{\beta_j\}_{j=0}^m$  to the matrix form:

$$\begin{aligned} {}_a^c D_s^{\beta_j} y(s) &= {}_a^c D_s^{\beta_j} [\mathbf{X}(s) \mathcal{C}(\alpha) \mathcal{A}] \\ &= s^{\omega_j^\beta - \beta_j} \mathbf{X}(s) \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \mathcal{C}(\alpha) \mathcal{A} \dots (18) \end{aligned}$$

where  $\mathcal{M}(\omega_j^\beta - \beta_j)$  is defined in equation (11).

We get the matrix relating to (19) by putting equation (3) into (15), and then substituting (15) and (18) into the Fredholm integral part  $IF(x, \{\beta_j\}_{j=0}^m)$  of equation (4):

$$\begin{aligned} IF(x, \{\beta_j\}_{j=0}^m) &= \left[ \sum_{j=0}^m \lambda_j \mathbf{X}(x) \mathcal{K}_t^j \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \right] \mathcal{C}(\alpha) \mathcal{A} \dots (20) \end{aligned}$$

### 3.3 Matrix Relation for the Mixed Conditions:

each  $k = 0, 1, \dots, \mu - 1$  and  $\mu = \max \{n, \omega_j^\beta : j = 0, 1, \dots, m\}$ , applying the relation (10) to a mixed condition of equation (2),  $\omega_j^\beta = [\beta_j]$  for all  $j =$

0,1,2,...,m. The steps to get the mixed condition matrix forms are as follows:

$$\begin{aligned} & \sum_{\ell=0}^{\mu-1} [\mathcal{G}_{k\ell} \mathbf{X}(a) [\mathcal{H}^T]^\ell \mathcal{C}(\alpha) \mathcal{A} \\ & + \mathcal{H}_{k\ell} \mathbf{X}(b) [\mathcal{H}^T]^\ell \mathcal{C}(\alpha) \mathcal{A}] \\ & = \vartheta_k, \text{ for all } k = 0, 1, \dots, \mu - 1 \end{aligned}$$

So that,

$$\begin{aligned} & \left[ \sum_{\ell=0}^{\mu-1} [\mathcal{G}_{k\ell} \mathbf{X}(a) + \mathcal{H}_{k\ell} \mathbf{X}(b)] [\mathcal{H}^T]^\ell \right] \mathcal{C}(\alpha) \mathcal{A} \\ & = \vartheta_k, \text{ for all } k \\ & = 0, 1, \dots, \mu - 1 \dots (21) \end{aligned}$$

#### 4- $\alpha$ -DICKSON MATRIX-COLLOCATION TECHNIQUE

We can now create it by inserting the matrix relations (13) and (20) into equation (4) using the basic matrix equation related to equation (1). So:

$$\begin{aligned} & \left[ p(x) \mathbf{X}(x) [\mathcal{H}^T]^n + \sum_{i=0}^{n-1} p_i(x) x^{\omega_i^\sigma - \sigma_i} \mathbf{X}(x) \mathcal{M}(\omega_i^\sigma - \sigma_i) [\mathcal{H}^T]^{\omega_i^\sigma} + p_n(x) \mathbf{X}(x) \right] \mathcal{C}(\alpha) \mathcal{A} \\ & = f(x) + \left[ \sum_{j=0}^m \lambda_j \mathbf{X}(x) \mathcal{K}_t^j \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \right] \mathcal{C}(\alpha) \mathcal{A} \dots (22) \end{aligned}$$

Applying the set of standard collocation points  $\{x_\ell | x_\ell = a + \left(\frac{b-a}{N}\right)\ell, \ell = 0, 1, \dots, N\}$ , the matrix

equation (22) forms the following system of equations, for each point  $x_\ell, \ell = 0, 1, \dots, N$ :

$$\begin{aligned} & \left[ p(x_\ell) \mathbf{X}(x_\ell) [\mathcal{H}^T]^n + \sum_{i=0}^{n-1} p_i(x_\ell) x_\ell^{\omega_i^\sigma - \sigma_i} \mathbf{X}(x_\ell) \mathcal{M}(\omega_i^\sigma - \sigma_i) [\mathcal{H}^T]^{\omega_i^\sigma} + p_n(x_\ell) \mathbf{X}(x_\ell) \right] \mathcal{C}(\alpha) \mathcal{A} \\ & = f(x_\ell) + \left[ \sum_{j=0}^m \lambda_j \mathbf{X}(x_\ell) \mathcal{K}_t^j \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \right] \mathcal{C}(\alpha) \mathcal{A} \dots (23) \end{aligned}$$

Alternatively, to put it briefly, the most important matrix equation is:

$$\begin{aligned} & \left[ \mathbf{P} \mathbf{X} [\mathcal{H}^T]^n + \sum_{i=0}^{n-1} \mathbf{P}_i \mathbf{y}_i \mathbf{X} \mathcal{M}(\omega_i^\sigma - \sigma_i) [\mathcal{H}^T]^{\omega_i^\sigma} + \mathbf{P}_n \mathbf{X} - \sum_{j=0}^m \lambda_j \mathbf{X} \mathcal{K}_t^j \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \right] \mathcal{C}(\alpha) \mathcal{A} \\ & = \mathbf{F} \dots (24) \end{aligned}$$

where

$$\begin{aligned} & \mathbf{y}_i \\ & = \begin{bmatrix} x_0^{(\omega_i^\sigma - \sigma_i)} & 0 & \dots & 0 \\ 0 & x_1^{(\omega_i^\sigma - \sigma_i)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_N^{(\omega_i^\sigma - \sigma_i)} \end{bmatrix}_{N+1 \times N+1} \\ & \mathbf{P} = \begin{bmatrix} p(x_0) & 0 & \dots & 0 \\ 0 & p(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p(x_N) \end{bmatrix}_{N+1 \times N+1} \end{aligned}$$

$$\begin{aligned} & \mathbf{P}_i \\ & = \begin{bmatrix} p_i(x_0) & 0 & \dots & 0 \\ 0 & p_i(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_i(x_N) \end{bmatrix}_{N+1 \times N+1}, \text{ For all } i \\ & = 0, 1, \dots, n \end{aligned}$$

while,  $\mathbf{y}_0 = [\mathbf{I}]_{(N+1) \times (N+1)}$  and  $\mathbf{P}_0 = [\mathbf{I}]_{(N+1) \times (N+1)}$  are the unit matrix, and

$$\mathbf{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}_{N+1 \times N+1}$$

Also, for each fractional order  $\mathfrak{T} = \{\sigma_i, \beta_j\}$  for each  $i = 0, 1, \dots, n - 1$  and  $j = 0, 1, \dots, m$ ; we are putting:

$$\mathcal{M}(\omega_{i \text{ or } j}^{\mathfrak{T}} - \mathfrak{T}) = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(\omega_{i \text{ or } j}^{\mathfrak{T}} - \mathfrak{T} + 1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(\omega_{i \text{ or } j}^{\mathfrak{T}} - \mathfrak{T} + 2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(\omega_{i \text{ or } j}^{\mathfrak{T}} - \mathfrak{T} + 3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(\omega_{i \text{ or } j}^{\mathfrak{T}} - \mathfrak{T} + N + 1)} \end{bmatrix}_{N+1 \times N+1}$$

$$\mathcal{H}^T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{N+1 \times N+1} \quad \mathcal{K}_t^j = \begin{bmatrix} {}_t\mathcal{K}_{00}^j & {}_t\mathcal{K}_{01}^j & \cdots & {}_t\mathcal{K}_{0N}^j \\ {}_t\mathcal{K}_{10}^j & {}_t\mathcal{K}_{11}^j & \cdots & {}_t\mathcal{K}_{1N}^j \\ \vdots & \vdots & \ddots & \vdots \\ {}_t\mathcal{K}_{N0}^j & {}_t\mathcal{K}_{N1}^j & \cdots & {}_t\mathcal{K}_{NN}^j \end{bmatrix}_{N+1 \times N+1}$$

where

$$\begin{aligned} {}_t\mathcal{K}_{mn}^j &= \frac{1}{m!n!} \frac{\partial^{m+n} k_j(0,0)}{\partial x^m \partial s^n} j \\ &= 0, 1, 2, \dots, m, n = 0, 1, 2, \dots, N \mathcal{Q}_j^\beta \\ &= \begin{bmatrix} q_{00}^j & q_{01}^j & \cdots & q_{0N}^j \\ q_{10}^j & q_{11}^j & \cdots & q_{1N}^j \\ \vdots & \vdots & \ddots & \vdots \\ q_{N0}^j & q_{N1}^j & \cdots & q_{NN}^j \end{bmatrix}_{N+1 \times N+1} \quad \text{for all } j \\ &= 0, 1, 2, \dots, m, \text{ and } \kappa, \rho = 0, 1, 2, \dots, N \end{aligned}$$

$$\mathcal{W} = \left[ \mathbf{P}\mathbf{X}[\mathcal{H}^T]^n + \sum_{i=0}^{n-1} \mathbf{P}_i \mathbf{y}_i \mathbf{X} \mathcal{M}(\omega_i^\sigma - \sigma_i) [\mathcal{H}^T]^{\omega_i^\sigma} + \mathbf{P}_n \mathbf{X} - \sum_{j=0}^m \lambda_j \mathbf{X} \mathcal{K}_t^j \mathcal{Q}_j^\beta \mathcal{M}(\omega_j^\beta - \beta_j) [\mathcal{H}^T]^{\omega_j^\beta} \right] \mathcal{C}(\alpha)$$

The set of  $(N+1)$  linear algebraic equations with unknown  $\alpha$ -Dickson coefficients is represented by equation (25):  $\mathcal{A} = [a_0, a_1, \dots, a_N]$ . The

where

$$\begin{aligned} \mu &= \max\{n, \omega_j^\beta\}, \text{ while } \omega_j^\beta - 1 < \beta_j \leq \omega_j^\beta, \omega_j^\beta \\ &= \lceil \beta_j \rceil \text{ for all } j = 0, 1, 2, \dots, m \end{aligned}$$

$$\begin{aligned} \mathbf{Y}_k &= \left[ \sum_{\ell=0}^{\mu-1} [\mathcal{G}_{k\ell} \mathbf{X}(a) + \mathcal{H}_{k\ell} \mathbf{X}(b)] [\mathcal{H}^T]^\ell \right] \mathcal{C}(\alpha) \\ &= [\mathcal{Y}_{k0} \mathcal{Y}_{k1} \mathcal{Y}_{k2} \cdots \mathcal{Y}_{kN}], k = 0, 1, \dots, \mu - 1. \end{aligned}$$

while,

$$\begin{aligned} \mathcal{Q}_j^\beta &= [q_{\kappa\rho}^{\beta,j}], q_{\kappa\rho}^{\beta,j} \\ &= \frac{b^{\omega_j^\beta - \beta_j + \kappa + \rho + 1} - a^{\omega_j^\beta - \beta_j + \kappa + \rho + 1}}{\omega_j^\beta - \beta_j + \kappa + \rho + 1} \end{aligned}$$

$$\mathcal{F} = [f(x_0) \ f(x_1) \ \cdots \ f(x_N)]_{N+1 \times 1}^T \quad \text{and}$$

$$\mathcal{A} = [a_0 \ a_1 \ \cdots \ a_N]_{N+1 \times 1}^T$$

while the matrices:

$\mathbf{y}_i, [\mathcal{H}^T]^n, \mathbf{X}, \mathcal{C}(\alpha), \mathbf{P}_i, \mathbf{P}_n, \mathcal{M}(\omega_i^\sigma - \sigma_i), \mathcal{M}(\omega_j^\beta - \beta_j), \mathcal{K}_t^j$  and  $\mathcal{Q}_j^\beta$  for all  $i = 0, 1, 2, \dots, n-1$  and  $n = 0, 1, \dots, N, j = 0, 1, \dots, m$ ; we have demonstrated in equation (24) that their dimensions are comparable to those of  $(N+1) \times (N+1)$ . As a result, the fundamental matrix equation (24) can be expressed as:

$$\mathcal{W}\mathcal{A} = \mathcal{F} \text{ or } [\mathcal{W} : \mathcal{F}] \quad \dots (25)$$

where

conditions' matrix form (21) can be shown as follows:

$$\mathbf{Y}_k \mathcal{A} = \vartheta_k \text{ or } [\mathbf{Y}_k : \vartheta_k] \text{ for all } k = 0, 1, \dots, \mu - 1 \quad \dots (26)$$

The solution of (1) under conditions (2) can be obtained by substituting any  $\mu$  rows of matrix (25)  $\mathcal{W}\mathcal{A} = \mathcal{F}$  for the row matrices (26) in the matrix. Replacing the final  $\mu$ -row of matrix (25) results in the following new augmented matrix for the preceding system:

$$[\mathcal{W}^* : \mathcal{F}^*] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & : & f(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & : & f(x_1) \\ w_{20} & w_{21} & w_{22} & \cdots & w_{2N} & : & f(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ w_{N-\mu,0} & w_{N-\mu,1} & w_{N-\mu,2} & \cdots & w_{N-\mu,N} & : & f(x_{N-\mu}) \\ \psi_{00} & \psi_{01} & \psi_{02} & \cdots & \psi_{0N} & : & \vartheta_0 \\ \psi_{10} & \psi_{11} & \psi_{12} & \cdots & \psi_{1N} & : & \vartheta_1 \\ \psi_{20} & \psi_{21} & \psi_{22} & \cdots & \psi_{2N} & : & \vartheta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ \psi_{\mu-1,0} & \psi_{\mu-1,1} & \psi_{\mu-1,2} & \cdots & \psi_{\mu-1,N} & : & \vartheta_{\mu-1} \end{bmatrix} \dots (27)$$

Note that we may write.  $\mathcal{A} = (\mathcal{W}^*)^{-1} \mathcal{F}^*$  if  $\text{rank } \mathcal{W}^* = \text{rank}[\mathcal{W}^* : \mathcal{F}^*] = N + 1$ . Therefore, for  $\det(\mathcal{W}^*) = 0$ , the  $\alpha$ -Dickson coefficients  $a_k$  ( $k = 0, 1, 2, \dots, N$ ) are uniquely defined by (27). When  $\text{rank}[\mathcal{W}^*] = \text{rank}[\mathcal{W}^* : \mathcal{F}^*] < N + 1$ , we might find particular solutions. Additionally, choose the  $N$  we specified, which has to be bigger than  $\mu$  (that is,  $N > \mu = \mu = \max \{n, \omega_j^{\beta}(\overline{0:m})\}$ ). If it is not, the matrix  $\mathcal{H}^T$  becomes a zero matrix, leading to a zero solution and making the suggested technique incapable of providing a solution.

## 5- NUMERICAL EXPERIMENTAL TESTS

This section demonstrates the viability of the present technique using numerical results from a few illustrative problems. Tables and figures also explain these results. We completed all tests using a computer program. We also provide a tabular version of the running time. We also provide a tabular version of the unni ie. Finally, for every test issue at the elected internal points, apply the absolute error function.  $(x_\ell) | y(x_\ell) - \hat{y}_N(x_\ell) |$ , for  $\dots, M$ . We also illustrate this with graphs.

**Test Problem 1.** Consider the mixed classical and fractional-order integro-differential equations of the Fredholm type with variable coefficients on the closed bounded interval  $[0, 1]$ :

$$\begin{aligned} y'(x) + x {}^C D_x^{0.1} y(x) - x y(x) \\ = f(x) \\ + \int_0^1 [e^x y(s) - s {}^C D_x^{0.6} y(s)] ds \end{aligned}$$

where

$$\begin{aligned} f(x) = \frac{2}{\Gamma(2.9)} x^{2.9} - \frac{1}{\Gamma(1.9)} x^{1.9} + \frac{2}{3.4 \Gamma(2.4)} \\ - \frac{1}{2.4 \Gamma(1.4)} - x^3 + x^2 + 2x + \frac{e^x}{6} \\ - 1 \end{aligned}$$

subject to the boundary source:  $2y(0) + y(1) = 0$ .

The exact solution of this equation is  $y(x) = x^2 - x$ . Here, from the example, we have:

$$\begin{aligned} n = m = 1; \sigma_0 = 0.1 \Rightarrow \omega_0^\sigma = [\sigma_0] = 1 \text{ and } \beta_0 = 0, \beta_1 = 0.6 \Rightarrow \omega_0^\beta = 0; \omega_1^\beta = [0.6] = 1. \text{ So, } \mu = \max \{1, 0, 1\} = 1. \\ \text{Also, } p(x) = 1, p_0(x) = x, p_1(x) = -x, \lambda_0 = 1, \lambda_1 = -1, \mathcal{K}_0(x, s) = e^x, \mathcal{K}_1(x, s) = s. \end{aligned}$$

Take  $N = 3$ , then the collocation points are  $\{x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1\}$ , and the basic matrix equation of the specified IFDEs-CF is derived from equation (24), written as:

$$\begin{aligned} \left[ P X [\mathcal{H}^T]^1 + P_0 \mathcal{Y}_0 X \mathcal{M}(\omega_0^\sigma - \sigma_0) [\mathcal{H}^T] \omega_0^\sigma + P_1 X - \lambda_0 X \mathcal{K}_0^0 \mathcal{Q}_0^\beta \mathcal{M}(\omega_0^\beta - \beta_0) [\mathcal{H}^T] \omega_0^\beta \right. \\ \left. - \lambda_1 X \mathcal{K}_1^1 \mathcal{Q}_1^\beta \mathcal{M}(\omega_1^\beta - \beta_1) [\mathcal{H}^T] \omega_1^\beta \right] \mathcal{C}(\alpha) \mathcal{A} = F \dots (28) \end{aligned}$$

where

$$\begin{aligned}
 X(x) &= [1 \quad x \quad x^2 \quad x^3], X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ X(x_2) \\ X(x_3) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
 P &= [I]_4, P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, P_1 \\
 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{K}_t^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \end{bmatrix}, \mathcal{K}_t^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{Q}_0^\beta \\
 &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, \mathcal{Q}_1^\beta \\
 &= 5 \begin{bmatrix} \frac{1}{7} & \frac{1}{12} & \frac{1}{17} & \frac{1}{22} \\ \frac{1}{12} & \frac{1}{17} & \frac{1}{22} & \frac{1}{27} \\ \frac{1}{17} & \frac{1}{22} & \frac{1}{27} & \frac{1}{32} \\ \frac{1}{22} & \frac{1}{27} & \frac{1}{32} & \frac{1}{37} \end{bmatrix} \\
 \mathcal{C}(\alpha) &= \begin{bmatrix} 2 & 0 & -2\alpha & 0 \\ 0 & 1 & 0 & -3\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, F \\
 &= \begin{bmatrix} -0.82938704 \\ -0.10641122 \\ 0.66652386 \\ 1.51171716 \end{bmatrix}
 \end{aligned}$$

After entering all of the previous matrices into matrix equation and computing the result, the augmented matrix of this basic matrix equation is:

$$[W:F] = \begin{bmatrix} -2 & 0.969608540827512 & 2\alpha + 0.140221497753233 & 0.207410916390433 - 2.90882562248253\alpha & : & -0.82938704 \\ -3.45679012345679 & 0.789910308256976 & 3.45679012345679\alpha + 0.683407303089107 & 0.445234317407345 - 2.36973092477093\alpha & : & -0.10641122 \\ -5.209876543209877 & 0.537263358021483 & 5.20987654320988\alpha + 1.20221015006176 & 1.34154811664862 - 1.61179007406445\alpha & : & 0.66652386 \\ -7.333333333333334 & 0.176029341841815 & 7.33333333333333\alpha + 1.67914397835308 & 2.92296290781557 - 0.5280880255254450\alpha & : & 1.51171716 \end{bmatrix}$$

Equation (26) forms the boundary condition rows, considering:  $Y_k \mathcal{A} = \vartheta_k$  or  $[Y_k : \vartheta_k]$ ;  $k = 0$ , or more precisely

$$[Y_0 : \vartheta_0] = [6 \ 1 \ 1 - 6\alpha \ 1 - 3\alpha : 0]$$

The augmented matrix for the system described in (27) is created based on specific conditions:

$$[W^*:F^*] = \begin{bmatrix} -2 & 0.969608540827512 & 2\alpha + 0.140221497753233 & 0.207410916390433 - 2.90882562248253\alpha & : & -0.82938704 \\ -3.45679012345679 & 0.789910308256976 & 3.45679012345679\alpha + 0.683407303089107 & 0.445234317407345 - 2.36973092477093\alpha & : & -0.10641122 \\ -5.209876543209877 & 0.537263358021483 & 5.20987654320988\alpha + 1.20221015006176 & 1.34154811664862 - 1.61179007406445\alpha & : & 0.66652386 \\ 6 & 1 & 1 - 6\alpha & 1 - 3\alpha & : & 0 \end{bmatrix}$$

By resolving this system, the  $\alpha$ -Dickson coefficient matrix  $\mathcal{A}$  if  $\alpha = 0$  is produced:

$$\mathcal{A} = [0.89991575 \quad -1.08823897 \quad 0.92562477 \quad -0.01649811]^T$$

Consequently, the  $\alpha$ -Dickson coefficients  $a_k$  ( $k = 0, 1, 2, 3$ ) uniquely determine; therefore, we obtain an approximate solution  $\hat{y}_3(x)$  in the form of the 3-truncated 0-Dickson expansion, is produced as

$$\begin{aligned}
 \hat{y}_3(x)|_{\alpha=0} &= 0.001991603368542x^3 \\
 &+ 0.998934186184855x^2 \\
 &- 1.0004391805954x \\
 &- 0.000162202985998228
 \end{aligned}$$

Using the same steps as before, we obtain the approximate solution for  $N = 9$  and  $N = 18$  when ( $\alpha$

= 0) effectively. After that, we run the general Python program created for this purpose:

$$\begin{aligned}
 \hat{y}_9(x)|_{\alpha=0} &= 5.3009384601 \times 10^{-8} x^9 \\
 &\quad - 1.353049576349 \times 10^{-7} x^8 \\
 &\quad + 1.613669975154 \times 10^{-7} x^7 \\
 &\quad - 1.107803940969 \times 10^{-7} x^6 \\
 &\quad + 4.628880762742 \times 10^{-8} x^5 \\
 &\quad - 1.180419924511 \times 10^{-8} x^4 \\
 &\quad + 1.544972677715 \times 10^{-9} x^3 \\
 &\quad + 0.999999998950 x^2 \\
 &\quad - 1.000000001584 x \\
 &\quad - 5.621048772162 \times 10^{-10} \\
 \hat{y}_{18}(x)|_{\alpha=0} &= 5.808059407448 \times 10^{-6} x^{18} \\
 &\quad - 4.922885731739 \times 10^{-5} x^{17} \\
 &\quad + 0.000192258246 x^{16} \\
 &\quad - 0.000458823309 x^{15} \\
 &\quad + 0.000748201062 x^{14} \\
 &\quad - 0.000883103382 x^{13} \\
 &\quad + 0.000779570958 x^{12} \\
 &\quad - 0.000524425884 x^{11} \\
 &\quad + 0.00027137753 x^{10} \\
 &\quad - 0.000108257609 x^9 \\
 &\quad + 3.314528322851 \times 10^{-5} x^8 \\
 &\quad - 7.700441074737 \times 10^{-6} x^7 \\
 &\quad + 1.330507993282 \times 10^{-6} x^6 \\
 &\quad - 1.654741474839 \times 10^{-7} x^5 \\
 &\quad + 1.403104905265 \times 10^{-8} x^4 \\
 &\quad - 7.338023602829 \times 10^{-10} x^3 \\
 &\quad + 1.000000000018 x^2 \\
 &\quad - 1.000000000001 x \\
 &\quad - 7.479025169221 \times 10^{-14}
 \end{aligned}$$

Table 1 compares the  $\alpha$ -Dickson polynomial solutions  $\hat{y}_N(x)$  and the exact solution  $y(x)$  of test problem 1 for  $N = 3, 9$  and  $18$ , respectively, with the aid of least-squares error and running time, and Table 2 compares the least-squares error for difference values of  $N$  and  $\alpha$ -Dickson's parameters. Figure 1(a and b) explains the analysis of the exact and our method results.

**Table 1. Numerical results for different values of  $x$  with  $N = 3, 9$  and  $18$ , for the  $(\alpha = 0)$ -Dickson polynomials**

| $x_i$       | Exact Solution<br>$y(x) = x^2 - x$ | Approximate Solutions $\hat{y}_N(x)$ |                                 |                                 |
|-------------|------------------------------------|--------------------------------------|---------------------------------|---------------------------------|
|             |                                    | $N = 3, \alpha = 0$                  | $N = 9, \alpha = 0$             | $N = 18, \alpha = 0$            |
| 0.0         | 0                                  | -1.62202986 × 10 <sup>-2</sup>       | -5.62104877 × 10 <sup>-2</sup>  | -7.47902517 × 10 <sup>-2</sup>  |
| 0.1         | -0.09                              | -9.02147876 × 10 <sup>-2</sup>       | -9.00000007 × 10 <sup>-2</sup>  | -9.00000000 × 10 <sup>-2</sup>  |
| 0.2         | -0.16                              | -1.60276739 × 10 <sup>-1</sup>       | -1.600000001 × 10 <sup>-1</sup> | -1.600000000 × 10 <sup>-1</sup> |
| 0.3         | -0.21                              | -2.10336107 × 10 <sup>-1</sup>       | -2.100000001 × 10 <sup>-1</sup> | -2.100000000 × 10 <sup>-1</sup> |
| 0.4         | -0.24                              | -2.40380943 × 10 <sup>-1</sup>       | -2.400000001 × 10 <sup>-1</sup> | -2.400000000 × 10 <sup>-1</sup> |
| 0.5         | -0.25                              | -2.50399296 × 10 <sup>-1</sup>       | -2.500000002 × 10 <sup>-1</sup> | -2.500000000 × 10 <sup>-1</sup> |
| 0.6         | -0.24                              | -2.40379218 × 10 <sup>-1</sup>       | -2.400000002 × 10 <sup>-1</sup> | -2.400000000 × 10 <sup>-1</sup> |
| 0.7         | -0.21                              | -2.10308758 × 10 <sup>-1</sup>       | -2.100000002 × 10 <sup>-1</sup> | -2.100000000 × 10 <sup>-1</sup> |
| 0.8         | -0.16                              | -1.60175967 × 10 <sup>-1</sup>       | -1.600000002 × 10 <sup>-1</sup> | -1.600000000 × 10 <sup>-1</sup> |
| 0.9         | -0.09                              | -8.99688959 × 10 <sup>-2</sup>       | -9.000000015 × 10 <sup>-2</sup> | -9.000000000 × 10 <sup>-2</sup> |
| 1.0         | 0.00                               | 3.24405972 × 10 <sup>-4</sup>        | 1.12421154 × 10 <sup>-9</sup>   | 1.53471602 × 10 <sup>-13</sup>  |
| LSE.        |                                    | 9.42860008 × 10 <sup>-7</sup>        | 2.36505762 × 10 <sup>-17</sup>  | 3.63384543 × 10 <sup>-25</sup>  |
| R. Time/Sec |                                    | 23.638810396                         | 18.190745115                    | 23.7096593379                   |

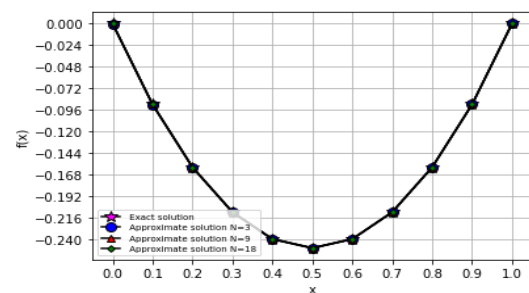


**Figure 1a. Analysis of the exact and our method results for test problem 1**

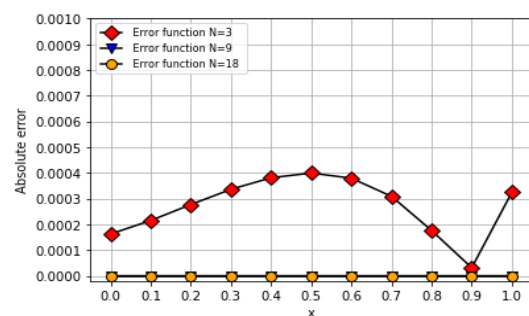


**Figure 1b. Absolute error of test problem 1**

**Table 2. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:**

| Values of Dickson's parameters ( $\alpha$ ) | Least squares error (LSE) for different inputs $N$ |                              |                              |
|---------------------------------------------|----------------------------------------------------|------------------------------|------------------------------|
|                                             | $N = 3$                                            | $N = 9$                      | $N = 18$                     |
| $\alpha = 0.8$                              | $9.42860008 \times 10^{-7}$                        | $2.36505379 \times 10^{-17}$ | $3.39154201 \times 10^{-23}$ |
| $\alpha = 10$                               | $9.42860008 \times 10^{-7}$                        | $2.36501452 \times 10^{-17}$ | $1.5546916 \times 10^{-28}$  |
| $\alpha = 20$                               | $9.42860008 \times 10^{-7}$                        | $2.36494586 \times 10^{-17}$ | $1.83362397 \times 10^{-28}$ |

**Test Problem 2.** Consider the Fredholm-type mixed classical and fractional-order integro-differential equations on the closed, bounded interval  $[0,1]$  that have variable coefficients:

$$\begin{aligned}
 y''(x) - \frac{x}{2} {}^C D_x^{1.2} y(x) + x^2 y(x) \\
 = f(x) - \frac{1}{4} \int_0^1 s \cos(x) y(s) ds \\
 + \frac{1}{2} \int_0^1 [2 \sin(x) + 1] {}^C D_s^{1.7} y(s) ds \\
 [PX[\mathcal{H}^T]^2 + P_0 \mathbf{y}_0 X \mathcal{M}(\omega_0^\sigma - \sigma_0)[\mathcal{H}^T]^{\omega_0^\sigma} + P_1 \mathbf{y}_1 X \mathcal{M}(\omega_1^\sigma - \sigma_1)[\mathcal{H}^T]^{\omega_1^\sigma} + P_2 X - \lambda_0 X \mathcal{K}_t^0 \mathcal{Q}_0^\beta \mathcal{M}(\omega_0^\beta - \beta_0)[\mathcal{H}^T]^{\omega_0^\beta} \\
 - \lambda_1 X \mathcal{K}_t^1 \mathcal{Q}_1^\beta \mathcal{M}(\omega_1^\beta - \beta_1)[\mathcal{H}^T]^{\omega_1^\beta}] \mathcal{C}(\alpha) \mathcal{A} = F
 \end{aligned}$$

After entering all of the matrices  $\{X, P, P_0, P_1, P_2, \mathbf{y}_0, \mathbf{y}_1, \mathcal{H}, \mathcal{M}, \mathcal{K}_t^0, \mathcal{K}_t^1, \mathcal{Q}_0^\beta, \mathcal{Q}_1^\beta, \mathcal{C}(\alpha)\}$  and  $F$  into the matrix equation above and computing the result using a Python program designed for this, we

$$[W^* : F^*] = \begin{bmatrix} 0.25 & 0.0833333333333333 & 1.20539037804054 - 0.25 \alpha & -1.06796907212104 - 0.25 \alpha & : & 3.9078378 \\ 0.4583333333333333 & 0.0414348782387062 & 0.625918330646555 - 0.458333333333333 \alpha & 0.192028931210135 - 0.124304634716119 \alpha & : & 1.97519971 \\ 2 & 0 & 1 - 2 \alpha & 1 & : & 4 \\ 2 & 0 & -2 \alpha & 0 & : & 1 \end{bmatrix}$$

After substituting each parameter mentioned above and executing the general Python program designed for this purpose with  $N = 3$ , and  $\alpha = 0$ . We get the approximate solution to the problem:

$$\begin{aligned}
 \hat{y}_3(x)|_{\alpha=0} &= 2.513775627929 \times 10^{-5} x^3 \\
 &+ 2.999974862244 x^2 \\
 &+ 2.000685765869 x + 1.0
 \end{aligned}$$

For  $N = 8$  and  $N = 16$ , when  $\alpha = 0$  we get the estimated solution of the problem by using the same

were

$$\begin{aligned}
 f(x) &= \frac{-3}{\Gamma(1.8)} x^{1.8} - \frac{6}{1.3 \Gamma(1.3)} \left[ \sin(x) + \frac{1}{2} \right] + 3x^4 \\
 &+ 2x^3 + x^2 + \frac{23}{48} \cos(x) + 6
 \end{aligned}$$

with the mixed conditions:  $y(1) - y'(0) = 4$  and  $y(0) = 1$ . While the exact solution is  $y(x) = 3x^2 + 2x + 1$ .

From the consideration of problem 2, the fractional orders:  $\sigma_0 = 0.2$ ,  $\sigma_1 = 1.2 \Rightarrow \omega_0^\sigma = [\sigma_0] = 1$ ,  $\omega_1^\sigma = [\sigma_1] = 2$ ;  $\beta_0 = 0$ ,  $\beta_1 = 1.7 \Rightarrow \omega_0^\beta = [1.7] = 2$ , so  $\mu = \max\{n, \omega_0^\beta, \omega_1^\beta\} = \max\{2, 0, 2\} = 2$ .  $p(x) = 1$ ,  $p_0(x) = 0$ ,  $p_1(x) = \frac{-1}{2}x$ ,  $p_2(x) = x^2$ .  $\lambda_0 = \frac{-1}{4}$ ;  $\lambda_1 = \frac{1}{2}$  and  $\mathcal{K}_0(x, s) = s \cos(x)$  with  $\mathcal{K}_1(x, s) = 2 \sin(x) + 1$ .

Take  $N = 3$ , the collocation points and the basic matrix formula of the provided IFDEs-CF are derived from equation (24), written as:

obtain the augmented matrix of this basic matrix equation with the boundary-condition rows. The augmented matrix for the system described in (27) is created based on specific conditions:

procedures above, then executing the General Python Program created for this reason:

$$\begin{aligned}
 \hat{y}_8(x)|_{\alpha=0} &= -1.426724585599 \times 10^{-6} x^8 \\
 &+ 2.971728079354 \times 10^{-6} x^7 \\
 &- 2.592876036208 \times 10^{-6} x^6 \\
 &+ 1.313196918972 \times 10^{-6} x^5 \\
 &- 2.886989659583 \times 10^{-7} x^4 \\
 &- 1.65332851124359 \times 10^{-8} x^3 \\
 &+ 3.00000003990787 x^2 \\
 &+ 1.999997328305 x + 1.0.
 \end{aligned}$$

$$\begin{aligned}\hat{y}_{16}(x)|_{\alpha=0} = & 8.453789007483 \times 10^{-9} x^{16} - 5.743589496226 \times 10^{-8} x^{15} \\ & + 1.764501614698 \times 10^{-7} x^{14} - 3.240020311467 \times 10^{-7} x^{13} \\ & + 3.959898171374 \times 10^{-7} x^{12} - 3.394521744511 \times 10^{-7} x^{11} \\ & + 2.094961584095 \times 10^{-7} x^{10} - 9.40296854664 \times 10^{-8} x^9 \\ & + 3.063071655653 \times 10^{-8} x^8 - 7.144982987147 \times 10^{-9} x^7 \\ & + 1.161867766646 \times 10^{-9} x^6 - 1.258873230092 \times 10^{-10} x^5 \\ & + 8.338307932265 \times 10^{-12} x^4 - 1.974526199189 \times 10^{-13} x^3 + 3.0 x^2 \\ & + 2.000000000003 x + 1.0\end{aligned}$$

The precise solution  $y(x)$  and the approximate solution  $\hat{y}_N(x)$  for  $N = 3, 8$  and  $16$  are compared in Table 3 based on running time and least-squares error, respectively, and Table 4 compares the least-squares error for difference values of  $N$  and  $\alpha$ -Dickson's parameters. Figure 2 (a and b) explains the analysis of the exact and our method results, with absolute errors.

**Table 3. Numerical results for different values of  $x$  with  $N = 3, 8$  and  $16$ , for the ( $\alpha = 0$ )-Dickson polynomials**

| $x_i$       | Exact Solution<br>$y(x) = 3x^2 + 2x + 1$ | Approximate Solutions $\hat{y}_N(x)$ |                              |                         |
|-------------|------------------------------------------|--------------------------------------|------------------------------|-------------------------|
|             |                                          | $N = 3, \alpha = 0$                  | $N = 8, \alpha = 0$          | $N = 16, \alpha = 0$    |
| 0.0         | 1                                        | 1                                    | 1                            | 1                       |
| 0.1         | 1.23                                     | 1.23006835                           | 1.22999973                   | 1.23                    |
| 0.2         | 1.52                                     | 1.52013635                           | 1.51999947                   | 1.52                    |
| 0.3         | 1.87                                     | 1.87020415                           | 1.8699992                    | 1.87                    |
| 0.4         | 2.28                                     | 2.28027189                           | 2.27999894                   | 2.28                    |
| 0.5         | 2.75                                     | 2.75033974                           | 2.74999867                   | 2.75                    |
| 0.6         | 3.28                                     | 3.28040784                           | 3.27999841                   | 3.28                    |
| 0.7         | 3.87                                     | 3.87047634                           | 3.86999815                   | 3.87                    |
| 0.8         | 4.52                                     | 4.5205454                            | 4.5199979                    | 4.52                    |
| 0.9         | 5.23                                     | 5.23061515                           | 5.22999763                   | 5.23                    |
| 1.0         | 6                                        | 6.00068577                           | 5.99999733                   | 6                       |
| LSE.        |                                          | $1.79366579 \times 10^{-6}$          | $2.70047677 \times 10^{-11}$ | $2.678 \times 10^{-17}$ |
| R. Time/Sec |                                          | 0.1094157695                         | 0.2239978313                 | 0.531                   |

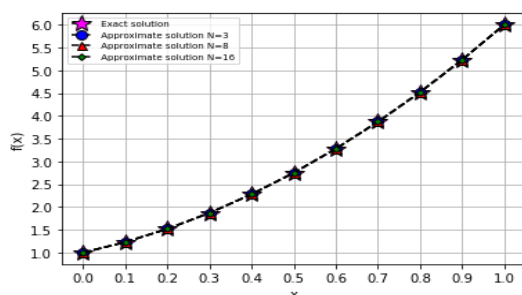


**Figure 2a. Analysis of the exact and our method results for test problem 2**

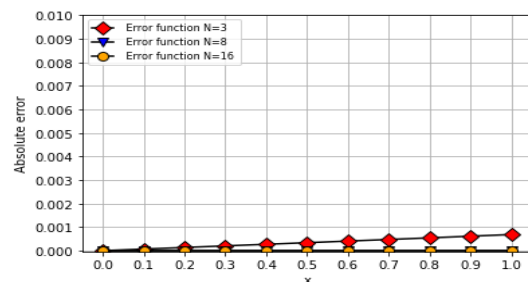


**Figure 2b. Absolute error of test problem 2**

**Table 4. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:**

| Value of Dickson's parameters | Least squares error (LSE) for different inputs $N$ |                              |                              |
|-------------------------------|----------------------------------------------------|------------------------------|------------------------------|
|                               | $N=3$                                              | $N=8$                        | $N=16$                       |
| $\alpha = 10$                 | $1.79366579 \times 10^{-6}$                        | $2.70047679 \times 10^{-11}$ | $5.13430066 \times 10^{-24}$ |
| $\alpha = 15$                 | $1.79366579 \times 10^{-6}$                        | $2.70047686 \times 10^{-11}$ | $1.23501105 \times 10^{-27}$ |
| $\alpha = 20$                 | $1.79366578 \times 10^{-6}$                        | $2.70047683 \times 10^{-11}$ | $3.9425164 \times 10^{-25}$  |

**Test Problem 3.** On the closed, bounded interval  $[0, 3]$ , take into consideration the Fredholm-type CF-IDEs with variable coefficients:

$$\begin{aligned}\frac{1}{6} y'''(x) + x {}^C_0 D_x^{0.2} y(x) - \frac{x}{4} {}^C_0 D_x^{0.5} y(x) + y(x) \\ = f(x) \\ + \frac{1}{100} \int_0^3 \left[ \sin\left(\frac{x}{2}\right) y(s) \right. \\ \left. - [2x - s] {}^C_0 D_s^{1.5} y(s) \right] ds\end{aligned}$$

where

$$\begin{aligned}f(x) = & \frac{6}{\Gamma(3.8)} x^{3.8} - \frac{3}{2 \Gamma(3.5)} x^{3.5} \\ & + \frac{6}{100 \Gamma(2.5)} \left[ \frac{2(3)^{2.5} x}{2.5} - \frac{3^{3.5}}{3.5} \right] \\ & + x^3 - \frac{1725}{1000} \sin\left(\frac{x}{2}\right)\end{aligned}$$

With the boundary conditions:  $y(0) + y''(0) = -1$ ,  $y(3) = 26$ ,  $y''(0) = 0$ . While the exact solution is  $y(x) = x^3 - 1$ .

From the consideration of problem 4, the fractional orders:  $\sigma_0 = 0.2$ ,  $\sigma_1 = 0.3$ ,  $\sigma_2 = 0.5 \Rightarrow \omega_0^\sigma = [\sigma_0] = 1$ ,  $\omega_1^\sigma = [\sigma_1] = 1$ ,  $\omega_2^\sigma = [\sigma_2] = 1$ ,  $\beta_0 = 0$ ,  $\beta_1 = 1.5 \Rightarrow \omega_1^\beta = [1.5] = 2$ , so  $\mu = \max\{n, \omega_0^\beta, \omega_1^\beta\} = \max\{3, 0, 2\} = 3$ . Also,  $p(x) = \frac{1}{6}$ ,  $p_0(x) = 1$ ,  $p_1(x) = 0$ ,  $p_2(x) = \frac{-x}{4}$ ,  $p_3(x) = 1$ .

$$\left[ pX[\mathcal{H}^T]^3 + p_0Y_0X\mathcal{M}(\omega_0^\sigma - \sigma_0)[\mathcal{H}^T]^{\omega_0^\sigma} + p_1Y_1X\mathcal{M}(\omega_1^\sigma - \sigma_1)[\mathcal{H}^T]^{\omega_1^\sigma} + p_2Y_2X\mathcal{M}(\omega_2^\sigma - \sigma_2)[\mathcal{H}^T]^{\omega_2^\sigma} + p_3X - \lambda_0X\mathcal{K}_t^0Q_0^\beta\mathcal{M}(\omega_0^\beta - \beta_0)[\mathcal{H}^T]^{\omega_0^\beta} - \lambda_1X\mathcal{K}_t^1Q_1^\beta\mathcal{M}(\omega_1^\beta - \beta_1)[\mathcal{H}^T]^{\omega_1^\beta} \right] \mathcal{C}(\alpha)\mathcal{A} = F$$

After substituting each parameter mentioned above and executing the general Python program designed for this purpose with  $N = 5$  and  $\alpha = 0$ , we obtain the approximate solution to the problem.

$$\begin{aligned} \hat{y}_5(x)|_{\alpha=0} &= 2.53893774886347 \times 10^{-7}x^5 \\ &- 8.63144992267004 \times 10^{-7}x^4 \\ &+ 1.00000015390038x^3 \\ &- 4.52849426513779 \times 10^{-16}x^2 \\ &+ 1.35441558019412 \times 10^{-6} \\ &- 0.999999999999999 \end{aligned}$$

For  $N = 11$  and  $N = 17$  when and  $\alpha = 0$ , we get the approximate solution of the problem by using the same procedures as above, then executing the most General Python Program created for this reason:

$$\begin{aligned} \hat{y}_{11}(x)|_{\alpha=0} &= 2.9338626748 \times 10^{-14}x^{11} \\ &- 1.016746573599 \times 10^{-12}x^{10} \\ &+ 5.135429241185 \times 10^{-12}x^9 \\ &- 9.2058525627988 \times 10^{-12}x^8 \\ &+ 1.736178106676 \times 10^{-11}x^7 \\ &- 1.514882356699 \times 10^{-11}x^6 \\ &- 2.897303209711 \times 10^{-11}x^5 \\ &- 1.102922560568 \times 10^{-10}x^4 \\ &+ 1.0000000000075x^3 \\ &- 2.785642900299 \times 10^{-16}x^2 \\ &+ 3.984965988645 \times 10^{-10}x \\ &- 0.999999999999999 \end{aligned}$$

and  $\hat{y}_{17}(x)|_{\alpha=0} = -1.654667030095 \times 10^{-15}x^{17} + 3.341203985699 \times 10^{-14}x^{16} - 3.069686072048 \times 10^{-13}x^{15} + 1.698781377278 \times 10^{-12}x^{14} - 6.318183129705 \times 10^{-12}x^{13} + 1.668184155342 \times 10^{-11}x^{12} - 3.219069991208 \times 10^{-11}x^{11} + 4.602994471688 \times 10^{-11}x^{10} -$

$$\lambda_0 = -\lambda_1 = 1/100. \quad \mathcal{K}_0(x, s) = \sin\left(\frac{x}{2}\right), \quad \mathcal{K}_1(x, s) = 2x - s.$$

Hence,  $\mu = 3$  and we can take  $N = 5$ ; the basic matrix equation and the collocation sites of the provided mixed-order integro-differential equation of Fredholm type are derived from equation (24), written as

$$\begin{aligned} &4.890002175032 \times 10^{-11}x^9 + \\ &3.831557755835 \times 10^{-11} - 2.170392427776 \times \\ &10^{-11}x^7 + 8.587808053958 \times 10^{-12}x^6 - \\ &2.314934160798 \times 10^{-12}x^5 - \\ &2.17406927383 \times 10^{-14}x^4 + \\ &1.000000000001x^3 - 1.011138077798 \times \\ &10^{-15}x^3 + 1.28662025479 \times 10^{-12}x - 1.0 \end{aligned}$$

The solution  $y(x)$  and the approximate solution  $\hat{y}_N(x)$  for  $N = 3, 8$  and  $16$  are compared in Table 5 based on running time and least-squares error, respectively, and Table 6 compares the least-squares error for difference values of  $N$  and  $\alpha$ -Dickson's parameters. Figure 3 (a and b) shows the analysis of the exact and our method results, with absolute errors.

**Table 5. Numerical results for different values of  $x$  with  $N = 3, 8$  and  $16$ , for the ( $\alpha = 0$ )-Dickson polynomials**

| $x_i$       | Exact solution | Approximate solutions $y_N(x)$ |                                |                                |
|-------------|----------------|--------------------------------|--------------------------------|--------------------------------|
|             |                | $N = 5, \alpha = 0$            | $N = 11, \alpha = 0$           | $N = 17, \alpha = 0$           |
| 0.0         | -1             | -1.0                           | -1.0                           | -1.0                           |
| 0.1         | -0.999         | -9.98999864 × 10 <sup>-1</sup> | -9.99000000 × 10 <sup>-1</sup> | -9.99000000 × 10 <sup>-1</sup> |
| 0.2         | -0.992         | -9.91999729 × 10 <sup>-1</sup> | -9.92000000 × 10 <sup>-1</sup> | -9.92000000 × 10 <sup>-1</sup> |
| 0.3         | -0.973         | -9.72999596 × 10 <sup>-1</sup> | -9.73000000 × 10 <sup>-1</sup> | -9.73000000 × 10 <sup>-1</sup> |
| 0.4         | -0.936         | -9.35999468 × 10 <sup>-1</sup> | -9.36000000 × 10 <sup>-1</sup> | -9.36000000 × 10 <sup>-1</sup> |
| 0.5         | -0.875         | -8.74999350 × 10 <sup>-1</sup> | -8.75000000 × 10 <sup>-1</sup> | -8.75000000 × 10 <sup>-1</sup> |
| 0.6         | -0.784         | -7.83999246 × 10 <sup>-1</sup> | -7.84000000 × 10 <sup>-1</sup> | -7.84000000 × 10 <sup>-1</sup> |
| 0.7         | -0.657         | -6.56999164 × 10 <sup>-1</sup> | -6.57000000 × 10 <sup>-1</sup> | -6.57000000 × 10 <sup>-1</sup> |
| 0.8         | -0.488         | -4.87999108 × 10 <sup>-1</sup> | -4.88000000 × 10 <sup>-1</sup> | -4.88000000 × 10 <sup>-1</sup> |
| 0.9         | -0.271         | -2.70999085 × 10 <sup>-1</sup> | -2.71000000 × 10 <sup>-1</sup> | -2.71000000 × 10 <sup>-1</sup> |
| 1.0         | 0              | 8.99064745 × 10 <sup>-2</sup>  | 3.30887984 × 10 <sup>-2</sup>  | 9.2481578 × 10 <sup>-2</sup>   |
| LSE.        |                | 4.66957325 × 10 <sup>-2</sup>  | 5.23143854 × 10 <sup>-2</sup>  | 4.6170544 × 10 <sup>-2</sup>   |
| R. Time/Sec |                | 0.0968568325                   | 0.2053394318                   | 0.40487670898                  |

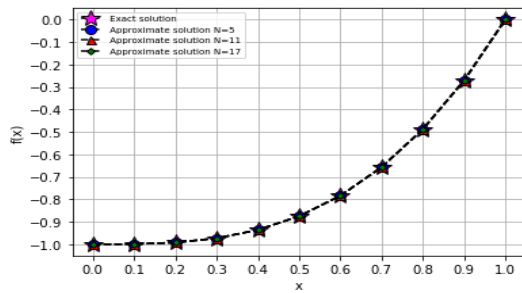


Figure 3a. Analysis of the exact and our method results for test problem 3

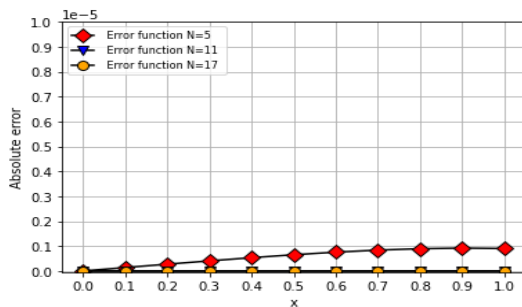


Figure 3b. Absolute error of test problem 3

Table 6. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:

| Value of Dickson's parameters | Least squares error (LSE) for different inputs $N$ |                              |                              |
|-------------------------------|----------------------------------------------------|------------------------------|------------------------------|
|                               | $N = 5$                                            | $N = 11$                     | $N = 17$                     |
| $\alpha = 1$                  | $4.66957327 \times 10^{-12}$                       | $5.23602057 \times 10^{-19}$ | $2.18256269 \times 10^{-23}$ |

$$f(x) = \sum_{\ell=0}^{\bar{N}} \left[ \frac{6x^{\ell+1.6}}{\Gamma(\ell+1.6)} + \frac{2.4}{\Gamma(\ell+1.2)} \left( \frac{1}{\ell+2.2} + \frac{3}{\ell+1.2} \right) \right] - 3e^xe^1 + 7e^x - 3$$

Hence,  $\mu = 1$ , the fundamental matrix equation of the given (Mixed-Order Integro-Differential

$$\begin{aligned} & [PX[\mathcal{H}^T]^1 + P_0Y_0X\mathcal{M}(\omega_0^\sigma - \sigma_0)[\mathcal{H}^T]^{\omega_0^\sigma} + P_2X - \lambda_0X\mathcal{K}_t^0Q_0^\beta\mathcal{M}(\omega_0^\beta - \beta_0)[\mathcal{H}^T]^{\omega_0^\beta} \\ & - \lambda_1X\mathcal{K}_t^1Q_1^\beta\mathcal{M}(\omega_1^\beta - \beta_1)[\mathcal{H}^T]^{\omega_1^\beta}] \mathcal{C}(\alpha)\mathcal{A} = F \end{aligned}$$

We select  $\bar{N} = 6$ , the estimated solution of the problem for  $N = 3, 5$  and  $11$  respectively, particularly when  $\alpha = 0$ :

$$\hat{y}_5(x)|_{\alpha=0} = 0.0331830819509585 x^5 + 0.115713052471747 x^4 + 0.509565731617031 x^3 + 1.49213981816473 x^2 + 3.00362993899182 x + 3.0$$

$$\begin{aligned} \text{and } \hat{y}_{11}(x)|_{\alpha=0} &= 0.000862503530018023 x^{11} - 0.00493209094929653 x^{10} + 0.0127424321429466 x^9 - 0.0198016533705852 x^8 + 0.0214406785912543 x^7 - 0.011450821413027 x^6 + \end{aligned}$$

|               |                              |                              |                              |
|---------------|------------------------------|------------------------------|------------------------------|
| $\alpha = 2$  | $4.66957325 \times 10^{-12}$ | $5.23669007 \times 10^{-19}$ | $2.29884455 \times 10^{-22}$ |
| $\alpha = 10$ | $4.66957342 \times 10^{-12}$ | $5.0910375 \times 10^{-19}$  | $1.60170184 \times 10^{-20}$ |

**Test Problem 4.** Consider the mixed classical and fractional-order IDEs of the Fredholm type with variable coefficients on the closed, bounded interval  $[0,1]$ :

$$\begin{aligned} & \frac{1}{3} y'(x) + 2x {}^C_0D_x^{0.4}y(x) + y(x) \\ &= f(x) + \int_0^1 [exp(x) + s] y(s) ds \\ & - \frac{8}{10} \int_0^1 [s + 3] {}^C_0D_x^{0.8}y(s) ds \end{aligned}$$

where

$$\begin{aligned} f(x) &= \sum_{\ell=0}^{\infty} \left[ \frac{6x^{\ell+1.6}}{\Gamma(\ell+1.6)} + \frac{2.4}{\Gamma(\ell+1.2)} \left( \frac{1}{\ell+2.2} + \frac{3}{\ell+1.2} \right) \right] - 3e^xe^1 + 7e^x - 3 \\ \text{The condition: } y(0) &= 3. \text{ While the exact solution is } y(x) = 3e^x. \\ \sigma_0 = 0.4 \Rightarrow \omega_0^\sigma &= [\sigma_0] = 1, \beta_0 = 0, \beta_1 = 0.8 \Rightarrow \omega_1^\beta = [\beta_1] = 1. \text{ So, } \mu = \max \{n, \omega_0^\beta, \omega_1^\beta\} = 1. \\ p(x) &= \frac{1}{3}, p_0(x) = 2x, p_1(x) = 1. \quad \mathcal{K}_0(x, s) = e^x + s, \mathcal{K}_1(x, s) = s + 3. \text{ And } \lambda_0 = 1, \lambda_1 = -0.8. \\ \text{Assume that we take } \bar{N} \text{ terms from the homogeneous part } f(x) : \end{aligned}$$

Equation of Fredholm Type) is derived from equation (24), written as:

$$\begin{aligned} \hat{y}_3(x)|_{\alpha=0} &= 0.801730029120443 x^3 + 1.18147978608646 x^2 + 3.15087173242313 x + 3.0 \end{aligned}$$

and

$$\begin{aligned} & 0.0334331066631315 x^5 + 0.121957704862877 x^4 + 0.500471802039465 x^3 + 1.50016966473797 x^2 + 2.99987668253542 x + 3.0 \end{aligned}$$

Tables (7, 8, 9 and 10) compare the exact solution.  $y(x)$  and the estimated solution  $\hat{y}_N(x)$  while we choose  $\bar{N} = 6, 10$  and 14. Also, we choose  $N = 3, 5$  and 11, each depending on the least-squares error and running time. While Figure 4 (a and b) explains the Analysis of the exact and our method results, with absolute errors for test problem 4 when  $\bar{N} = 6$ .

**Table 7 compares the exact solution.  $y(x)$  and the estimated solution  $\hat{y}_N(x)$  for  $N = 3, 5$  and 11, separately, evaluated in terms of least-squares error and running time. when  $\bar{N} = 6$ .**

| $x_i$       | Exact solution | $\bar{N} = 6$                  |                             |                             |
|-------------|----------------|--------------------------------|-----------------------------|-----------------------------|
|             |                | Approximate solutions $y_N(x)$ |                             |                             |
|             |                | $N = 3, \alpha = 0$            | $N = 5, \alpha = 0$         | $N = 11, \alpha = 0$        |
| 0.0         | 3              | 3                              | 3                           | 3                           |
| 0.1         | 3.31551275     | 3.3277037                      | 3.31580586                  | 3.31550236                  |
| 0.2         | 3.66420827     | 3.68384738                     | 3.66468387                  | 3.66419123                  |
| 0.3         | 4.04957642     | 4.07324141                     | 4.05015775                  | 4.04955538                  |
| 0.4         | 4.47547409     | 4.50069618                     | 4.4761086                   | 4.47545059                  |
| 0.5         | 4.94616381     | 4.97102207                     | 4.94681468                  | 4.9461386                   |
| 0.6         | 5.4663564      | 5.48902945                     | 5.46699122                  | 5.46632956                  |
| 0.7         | 6.04125812     | 6.05952871                     | 6.0418303                   | 6.04122876                  |
| 0.8         | 6.67662279     | 6.68733022                     | 6.67704059                  | 6.67658801                  |
| 0.9         | 7.37880933     | 7.37724438                     | 7.37888723                  | 7.37876199                  |
| 1.0         | 8.15484549     | 8.13408155                     | 8.15423162                  | 8.15477001                  |
| LSE.        |                | 0.00374455                     | $2.76410777 \times 10^{-6}$ | $1.27598754 \times 10^{-8}$ |
| R. Time/Sec |                | 0.08797097206                  | 0.0663876534                | 0.223850965                 |

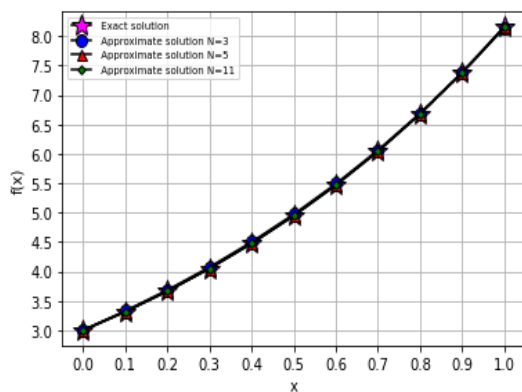


**Figure 4a1. Analysis of the exact and our method results for test problem 4 when  $\bar{N} = 6$**

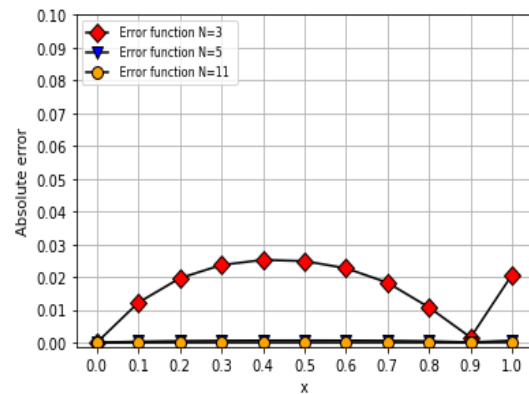


**Figure 4b1. Absolute error of test problem 4 when  $\bar{N} = 6$**

**Table 8. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:**

| Value of Dickson's parameters | $\bar{N} = 6$                                      |                             |                             |
|-------------------------------|----------------------------------------------------|-----------------------------|-----------------------------|
|                               | Least-squares error (LSE) for different inputs $N$ |                             |                             |
|                               | $N = 3$                                            | $N = 5$                     | $N = 11$                    |
| $\alpha = 10^2$               | 0.003744                                           | $2.7641078 \times 10^{-6}$  | $1.27616577 \times 10^{-8}$ |
| $\alpha = 10^4$               | 0.003744                                           | $2.76385978 \times 10^{-6}$ | $1.10485803 \times 10^{-8}$ |
| $\alpha = 4 \times 10^6$      | 0.003744                                           | 0.0002136                   | $3.60298368 \times 10^{-6}$ |

**Table 9. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:**

| Value of Dickson's parameters | $\bar{N} = 10$                                     |                              |                              |
|-------------------------------|----------------------------------------------------|------------------------------|------------------------------|
|                               | Least-squares error (LSE) for different inputs $N$ |                              |                              |
|                               | $N = 3$                                            | $N = 5$                      | $N = 11$                     |
| $\alpha = 10^2$               | 0.003751                                           | $2.94640593e \times 10^{-6}$ | $6.00972747 \times 10^{-17}$ |
| $\alpha = 3 \times 10^4$      | 0.003751                                           | $2.94744089 \times 10^{-6}$  | $2.13912108 \times 10^{-8}$  |
| $\alpha = 10^6$               | 0.003751                                           | $2.87453974 \times 10^{-6}$  | $6.21709563 \times 10^{-8}$  |

**Table 10. Comparison of least-squares error for various values of  $N$  and  $\alpha$ -Dickson's parameters:**

| Value of Dickson's parameters | $\bar{N} = 14$                                     |                             |                              |
|-------------------------------|----------------------------------------------------|-----------------------------|------------------------------|
|                               | Least-squares error (LSE) for different inputs $N$ |                             |                              |
|                               | $N = 3$                                            | $N = 5$                     | $N = 11$                     |
| $\alpha = 5 \times 10^2$      | 0.003751                                           | $2.94642611 \times 10^{-6}$ | $3.14771265 \times 10^{-16}$ |
| $\alpha = 5 \times 10^3$      | 0.003751                                           | $2.94633908 \times 10^{-6}$ | $1.44628387 \times 10^{-13}$ |
| $\alpha = 5 \times 10^4$      | 0.003751                                           | $2.94652866 \times 10^{-6}$ | $4.08523076 \times 10^{-7}$  |

## 6- CONCLUSIONS

Analytical solutions to linear mixed-order (classical and fractional) Fredholm integro-differential equations are often difficult. Therefore, approximate solutions are often necessary. This new methodology has enabled the discovery of approximate solutions for various applications. The most noteworthy advantage of the present methodology is Dickson Interpolations, which are readily implemented using computer programs. The existence of parameter- $\alpha$  is required for the accuracy of solutions when employing a computer program in addition to the existing method. The results are shown with examples in Tables and Figures. Tables and numbers make it abundantly evident that as  $N$  increases, the error rate reduces and the answer becomes closer to the correct solution. One significant benefit of the approach is that it uses computer code developed in Python v3.8.8 (2021) to enable reasonably speedy computation of the solution's Dickson coefficients.

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