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On $\mathcal{S}w$ -Topological Groups

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ABSTRACT

This paper investigates the class of $\mathcal{S}w$-topological groups, which are determined by $\mathcal{S}w$-open sets and $\mathcal{S}w$-continuity. It is demonstrated that these groups are a generalization of topological groups and differ from several alternative concepts of semi-topological groups in the literature. These ideas are reinforced with counterexamples. We describe some fundamental results and uses of $\mathcal{S}w$-topological groups. Investigations are conducted into the similarities and differences with topological groups.

Keywords: Topological group, Sw-topological group, Sw-open, Sw-continuity
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حول $\mathcal{S}w$ -المجموعات الطوبولوجية

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الملخص

يتناول هذا البحث فئة من $\mathcal{S}w$-المجموعات الطوبولوجية، التي تُحدد من خلال الخصائص $\mathcal{S}w$-المفتوحة و$\mathcal{S}w$- استمرارية وقد أظهرت الدراسات أن تمثل تعميماً للمجموعات مجموعات الطوبولوجية، وتختلف عن عدد من المفاهيم الأدبية البديلة للمجموعات شبه الطوبولوجية. تدعم هذه الأفكار أمثلة مضادة، كما قمنا بوصف بعض النتائج والاستخدامات الأساسية للمجموعات الطوبولوجية. يتم أيضاً استكشاف أوجه التشابه والاختلاف بينها وبين المجموعات الطوبولوجية.

INTRODUCTION

When a set has both algebraic and topological structures, it is natural to study how these interact. Usually, one structure serves as the foundation, while the other guides behavior. This is true in the study of topological groups: both the inverse and the multiplication mappings are continuous. Similar conditions apply to topological vector spaces, topological rings, and so forth. The study of topological groups has benefited greatly from the contributions of numerous mathematicians. Early proponents of the theory of topological groups include A.D. Alexandroff, N. Bourbaki, M.I. Graev, S. Kakutani, E. van Kampen, A.N. Kolmogorov, A.A. Markov, Pontryagin, and others.

This kind of research in the context of topological groups has led to the investigation of semi-topological groups.[1], topological structures of transformation groups[2], S-topological groups[3]Almost paratopological groups [4], β-Ideal topological group (5), S-Topological Transformation Group [6], Isotropy group on some topological transformation group structures[7], irresolute topological groups [8], p-topological groups[9], Relations on topologized groups[10], which began in the 1930s and 1950s. Finding the circumstances in which these classes of groups are topological groups is a logical suggestion. We direct the reader to two great sources: A new notion of open sets, known as Sw −open sets, was defined by Abdulla L. S. and Khalaf A. B. in 2007 [11]. Tkachenko's survey paper [12] and Arhangel'skii and Tkachenko's monograph [13], along with the references therein. The concept of $\mathcal{S}w$-topological groups have already been introduced in the literature [2]. We use standard information and notation for Semi-open sets in topological spaces, as defined by N. Levine. [14] In 1963 and in 2007, Abdulla L. S. and Khalaf A. B. have defined a new concept of open sets, which are called. $\mathcal{S}w$-open sets [11]Many mathematicians have investigated various ideas and expanded upon them through the use of S_p-open sets (15), π-continuity in [16], semi ω-open sets [17], ic-open sets [18]. The $\mathcal{S}w$-open: A subset A of a topological space (X, τ) together with the empty set is said to be somewhat open (briefly $\mathcal{S}w$-open) (11) if $t(A) \neq \emptyset$. The collection of every $\mathcal{S}w$-open set in X is represented by $\mathcal{S}w$-$O(X)$. A $\mathcal{S}w$-closed is the complement of a $\mathcal{S}w$-open set, the intersection of all $\mathcal{S}w$-closed subsets of X that contain A is the $\mathcal{S}w$-closure of $A \subset$ X, represented by $\mathcal{S}w$-$Cl(A)$ [11]. Let us state that x ∈ $\mathcal{S}w$-$Cl(A)$ if and only if $U \cap A \neq \emptyset$ for each $\mathcal{S}w$-open set U that contains x.

Every open (closed) set is $\mathcal{S}$w-open ($\mathcal{S}$w-closed); The intersection of two $\mathcal{S}$w-open sets need not be $\mathcal{S}$w-open ; if $A \subset X$ and $B \subset Y$ are $\mathcal{S}$w-open in spaces X and Y, then $A \times B$ is $\mathcal{S}$w-open in the product space $X \times Y$, the union of any collection of $\mathcal{S}$w-open sets are also a $\mathcal{S}$w-open set. Basic properties of $\mathcal{S}$w-closure, $\mathcal{S}$w-open sets, and $\mathcal{S}$w-closed sets in [11].

Definition 1.1. [11] Assume that X and Y are topological spaces. [11] If, $f(V) \in \mathcal{S}$w-$O(X)$ for every open set V in Y, then a mapping: $X \rightarrow Y$ is $\mathcal{S}$w-continuous. In [19] If the image of any open set in Y is $\mathcal{S}$w-closed in X, then a function $f: X \longrightarrow Y$ is contra- $\mathcal{S}$w-continuous.

Consider that if a set A ∈ $\mathcal{S}w$-$O(X)$ such that x ∈ A ⊂ U, then a set U ⊂ X is a *$\mathcal{S}w$-neighborhood* of a point $x \in X$ If and only if A is a $\mathcal{S}$ *w-neighborhood* of each point in a set $A \subset X$, then A is $\mathcal{S}w$-open in X.

Definition 1.2. [11] If a *$\mathcal{S}w$-neighborhood* U of a point x is $\mathcal{S}w$-open then it said to be an $\mathcal{S}w$-open neighborhood of x.

Remark 1.3. [12, 13] (The multiplication mapping is continuous in each variable separately. The inverse mapping is continuous. **Definition 1.4**. l space, A function$f: X \longrightarrow Y$ is called $\mathcal{S}w$-irresolute at $x \in X$ If for every $\mathcal{S}w$-open set U in Y containing $f(x)$ There exists a $\mathcal{S}w$-open set G in X containing x such that $f(G) \subset U$. A function $f: X \longrightarrow Y$ is $\mathcal{S}w$-irresolute if it is a $\mathcal{S}w$-irresolute at each $x \in X$.

Theorem 1.5. ([11]) A function $f: X \longrightarrow Y$ is $\mathcal{S}w$-irresolute if and only if $f^{-1}(U)$ is $\mathcal{S}w$ -open for every $\mathcal{S}w$ -open set U *in* Y.

$\mathcal{S}w$-Topological Group

Definition 2.1. The triple $(\mathbb{G},*,\mathcal{T})$ is considered a $\mathcal{S}w$-topological group if $(\mathbb{G},*)$ is a group, $(\mathbb{G},\mathcal{T})$ is a topological space, and a multiplication mapping m: $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$, defined by $m(x,y) = x * y$, where $x * y \in \mathbb{G}$, is $\mathcal{S}w$-continuous, and the inverse mapping i: $\mathbb{G} \to \mathbb{G}$, defined by $i(x) = x^{-1}$, $x \in \mathbb{G}$, is $\mathcal{S}w$-continuous.

Theorem 2.2. The triple ($\mathbb{G}$,$*$, $\mathcal{T}$) is a $\mathcal{S}$ *a topological* group if and only if for each neighborhood $\mathcal{W}$ of $x * y^{-1}$, where $x.y$ in $\mathbb{G}$, has $\mathcal{S}w$- open neighbourhoods $\mathcal{U}$ of x and $\mathcal{V}$ of y such that $\mathcal{U}*\mathcal{V}^{-1} \subset \mathcal{W}$.

Remark 2.3. The definition implies that all topological groups are both semi-topological groups [20], and $\mathcal{S}w$-topological groups. The examples that follow demonstrate that the opposite is not true.

Example 2.4. Assume that the two-element (cyclic) group $\mathbb{G} = Z_2 = \{0, 1\}$ has the multiplication mapping m = +2, the standard addition modulo 2. Give $\mathbb{G}$ the Sierpiński topology, which is τ = {∅, {0}, }.It is clear that $\mathbb{G}$ is a semi-topological group and $\mathcal{S}w$-topological group, but not a topological group.

Example 2.5. The topology $\mathcal{T}$ = {∅, $\mathbb{G}$, {1}, {1, 3, 5}}, in the set $\mathbb{G}$ = {1, 3, 5, 7} which is an abelian group when m = {8 - the standard multiplication modulo 8 is used. This is a semi-topological group and $\mathcal{S}w$-topological group but not a topological group.

Example 2.6. Let $\mathbb{G} = \{0.1.2.3\}$ with addition modulo 4, with topology $\mathcal{T}$ ={∅, {0,2}, $\mathbb{G}$} which is a $\mathcal{S}w$-topological group, but not a semi-topological group.

Theorem 2.7. States that if $(\mathbb{G},*.\mathcal{T})$ is a $\mathcal{S}w$-topological group:

(1) A ∈ $\mathcal{S}w$O($\mathbb{G}$) if and only if A^{-1} ∈ $\mathcal{S}w$O($\mathbb{G}$).

(2) if A ∈ $\mathcal{S}w$O($\mathbb{G}$) and B ⊂ $\mathbb{G}$, then both A∗B and B∗A are in $\mathcal{S}w$O($\mathbb{G}$).

Proof. (1) since A∈ $\mathcal{S}w$O($\mathbb{G}$) so A is a $\mathcal{S}w$-open set and Int(A)≠ ∅, so there exists an open p∈A is called an interior point of A, and there exists an open set containing A. $\mathcal{U} \in \mathbb{G}$such that $p \in \mathcal{U} \subseteq A$ and $p^{-1} \in \mathcal{U}^{-1} \subseteq A$, the conclusion follows since $Int(A^{-1})$≠ ∅. This A^{-1} is a $\mathcal{S}w$-topological group.
(2) Let $b \in B$ and $\in A * b$, $c = a * b$ for same $a \in A$, So $A = (A * b) * b^{-1}$. Since there exist open neighborhoods $\mathcal{U}_a$ *and* $\mathcal{U}_b$ such that $\mathcal{U}_a * b^{-1} \subset \mathcal{U}_b * {\mathcal{U}_a}^{-1} \subset A$, We have $\mathcal{U}_a \subset A * b$ thus $A * b$ is $\mathcal{S}w$-open set, So $A * b \in \mathcal{S}wO(\mathbb{G})$ *by* [11] So we get that $A * B \in \mathcal{S}wO(\mathbb{G})$.

Corollary 2.8. Let $(\mathbb{G}.*,\mathcal{T})$ is a $\mathcal{S}w$-topological group, then:

(1) $A \in \mathcal{S}w$C($\mathbb{G}$) if and only if $A^{-1} \in \mathcal{S}w$C($\mathbb{G}$);

(2) if $A \in \mathcal{S}w$C($\mathbb{G}$) and $B \subset \mathbb{G}$, then both $A * B$ and $B * A$ are in $\mathcal{S}w$C($\mathbb{G}$).

Definition 2.9. A is said to be symmetric if $= A^{-1}$, where A is a subset of a $\mathcal{S}w$-topological group.

Definition 2.10. Let X and Y are two $\mathcal{S}w$-topological groups, A function $f: X \longrightarrow Y$ is pre-$\mathcal{S}$w-open if for every $\mathcal{S}w$-open set A of X, the set $f(A)$ is $\mathcal{S}w$-open in Y.

Definition 2.11. Given two $\mathcal{S}w$-topological groups X and Y, A function $f: X \longrightarrow Y$ $\mathcal{S}w$-homeomorphism if it is pre- $\mathcal{S}w$-open, $\mathcal{S}w$-irresolute, and bijective.

If a bijective function f:$X \to Y$ is $\mathcal{S}w$-continuous and pre- $\mathcal{S}w$-open., then it is a $\mathcal{S}w$-homeomorphism.

Theorem 2.12. Every translation left(right) l_g: $\mathbb{G} \to \mathbb{G}$ (r_g: $\mathbb{G} \to \mathbb{G}$), is a $\mathcal{S}w$-homeomorphism, in a $\mathcal{S}w$-topological group $(\mathbb{G},*.\mathcal{T})$.

. or left translations only; the statement will; the other translations follow similarly. First, we prove that l_x is a bijection. Suppose an element of duble-struk cap G, then the element superscript base, y a., ed base, to the, minus 1 end supescramapse,toip ma tupscrptasoe hay $\in \mathbb{G}$then thement ya^{-1}maps to y, so r_a is surjective. Assume that $l_x(x) = l_x(y)$, which means that $xa = ya$.We multiply by a^{-1} on the right to get $x = y$, so l_x is injective. We directly demonstrate the $\mathcal{S}w$-continuity of the translation l_x for each $x \in \mathbb{G}$. Let W be a $\mathcal{S}w$-open neighbourhood of $l_x(y = x * y = x * (y^{-1})^{-1})$, and let y be an arbitrary element in $\mathbb{G}$. According to the definition of $\mathcal{S}w$-topological groups, U and V are $\mathcal{S}w$-open sets that contain x and y^{-1}, respectively, such that $U * V^{-1} \subset W$. Specifically, we have $x * V^{-1} \subset W$. Theorem 2.7 states that the set V^{-1} is a $\mathcal{S}w$-open neighbourhood of, indicating that x is $\mathcal{S}w$-continuous at y in the final inclusion. l_x is a $\mathcal{S}w$-continuous on $\mathbb{G}$,since $y \in \mathbb{G}$ was an arbitrary element in $\mathbb{G}$. It is now established that l_x is pre-$\mathcal{S}w$-open. Let A be a set in $\mathbb{G}$ that is $\mathcal{S}w$-open. According to Theorem 2.7, l_x is a $\mathcal{S}w$-open mapping since the set l_x(A) = $x * A = \{x\} * A$ is pre-$\mathcal{S}w$-open in $\mathbb{G}$.

Definition 2.13. A $\mathcal{S}w$-homeomorphism f of a $\mathcal{S}w$-topological space $\mathbb{G}$ onto itself such that $f(x) = y$ exists for any $x * y \in \mathbb{G}$. This space is said to be $\mathcal{S}w$-homogeneous.

Corollary 2.14. All $\mathcal{S}w$-topological groups $\mathbb{G}$ are $\mathcal{S}w$-homogeneous spaces.

Theorem 2.15. Assume that H is a subgroup of $\mathbb{G}$ and that ($\mathbb{G}$,∗.$\mathcal{T}$) is a $\mathcal{S}w$-topological group, H is $\mathcal{S}w$-open in $\mathbb{G}$ if it contains a non-empty $\mathcal{S}w$-open set.

Definition 2.16. A $\mathcal{S}A$ *topological* subgroup is a subset of $\mathcal{S}w$-topological group that itself is a $\mathcal{S}w$-topological group.

Theorem 2.17. Each subgroup H of a $\mathcal{S}w$-topological group ($\mathbb{G}$,∗.$\mathcal{T}$) which is open, then it is also a $\mathcal{S}w$-topological group.

Theorem 2.18. Every open subgroup of $\mathbb{G}$ is $\mathcal{S}w$-closed in $\mathbb{G}$, when ($\mathbb{G}$,∗.$\mathcal{T}$) a $\mathcal{S}w$-topological group.

Proof. Let H be an open subgroup of $\mathbb{G}$. Then every left coset $x * H$ of H is $\mathcal{S}w$-open because l_x is a pre- $\mathcal{S}w$-open mapping. Thus, Y = $S_{x \in G/H}\ x * H$ is also $\mathcal{S}w$-open as a union of $\mathcal{S}w$-open sets. Then H = $\mathbb{G}$ / Y and so H is $\mathcal{S}w$-closed.

We know that if a topological group. $\mathbb{G}$ has a subgroup H, then Cl(H) is a subgroup of $\mathbb{G}$. Now we study it to $\mathcal{S}w$-topological groups.

Example 2.19. Consider the topology $\mathcal{T}$ = {∅, G, {1}, {1, 3}} in the set $\mathbb{G}$ = {1, 3, 5, 7}, that is an abelian group when m = {8 - the standard multiplication modulo 8} is used Which is $\mathcal{S}w$-topological group. If we choose H = {1,3}, it is clear that H is a subgroup of the group. $\mathbb{G}$, and Cl(H) = G, which is a subgroup of G. $\mathcal{S}w$-Cl(H) = $\mathbb{G}$ is also a subgroup of $\mathbb{G}$.

Theorem 2.20. Assume that $f: \mathbb{G} \to H$ is a $\mathcal{S}w$-homeomorphism of $\mathcal{S}w$-topological groups. f is $\mathcal{S}w$-irresolute (and hence $\mathcal{S}w$-continuous) on $\mathbb{G}$ if it is $\mathcal{S}w$-irresolute at the neutral element e_G of $\mathbb{G}$.

Proof. Assume that x is in $\mathbb{G}$. Assume that W is a $\mathcal{S}w$-open neighbourhood in H where $y = f(x)$. Given that the left translations in H are $\mathcal{S}w$-continuous mappings, the neutral element e of H has a $\mathcal{S}w$-open neighbourhood V such that l_y(V) = y ∗ V ⊂ W. Given that f is $\mathcal{S}w$-irresolute at $e_{\mathbb{G}}$, there must be a $\mathcal{S}w$-open set U ⊂ $\mathbb{G}$ that contains $e_{\mathbb{G}}$ such that f(U) ⊂ V. The set $x * U$ is a $\mathcal{S}w$-open neighbourhood of x since $l_x: \mathbb{G} \to \mathbb{G}$ is a pre- $\mathcal{S}w$-open mapping. It follows that $f(x * U) = f(x) * f(U) = y * f(U) \subset y * V \subset$ W.

Because x was an arbitrary element in $\mathbb{G}$, f is $\mathcal{S}w$-irresolute (and so $\mathcal{S}w$-continuous) at that point and accordingly on $\mathbb{G}$.

Proposition 2.21. Assume that $f: \mathbb{G} \to H$ is a $\mathcal{S}w$-homeomorphism of $\mathcal{S}w$-topological groups. The multiplication map f is $\mathcal{S}w$-irresolute if and only if it is a $\mathcal{S}w$- continuous at each point.

Proof. Let f is a $\mathcal{S}w$-irresolute so it is l_a(resp., r_a) is a $\mathcal{S}w$-continuous at each point by Theorem 2.12 for each $a \in X$ and conversely by Theorem 2.20, f *is a* $\mathcal{S}w$-irresolute.

Theorem 2.22. A $\mathcal{S}w$-topological group ($\mathbb{G}$,∗.$\mathcal{T}$) with base βe at the identity element e. This group has a symmetric $\mathcal{S}w$-open neighbourhood V such that $V * V \subset U$ for every $U \in \beta e$. Then, at e, $\mathbb{G}$ satisfies the $\mathcal{S}w$-regularity assumption.

Proo. Suppose that U is an open set that contains the identity. Therefore, a symmetric $\mathcal{S}w$-pen neighborhood V of e that satisfies V ∗ V ⊂ is assumed. , equation script cap w-open n, equation script cap Sw-open, equation script cap Swope, equation script cap Sw-open n, equation script cap

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Sw-open neighborhood to demonstrate that $\mathcal{S}w$-l(V) ⊂ U. Let x be a member of $\mathcal{S}w$-Cl(V). As a $\mathcal{S}w$-open neighbourhood of x, the set $x * V$ suggests that $x * V \cap V \neq \emptyset$. It follows that there are points $a * b \in$ V such that $b = x * a$, that is, $x = b * a^{-1} \in V * V^{-1}$ = V ∗V ⊂ U.

Theorem 2.23. In a $\mathcal{S}w$-topological group ($\mathbb{G}$,∗.$\mathcal{T}$) let A and B There are two subsets. Then:

(1) $\mathcal{S}w$-$Cl(A)$∗$\mathcal{S}w$-$Cl(B) \subset Cl(A * B)$.

(2) $(\mathcal{S}w\text{-Cl}(A))^{-1} \subset Cl(A^{-1})$.

Theorem 2.24. Let ($\mathbb{G}$,∗.$\mathcal{T}$) be a $\mathcal{S}a$ *topological* group and the multiplication map f is $\mathcal{S}w$-continuous. For any $A.B \subseteq \mathbb{G}$ and $a \in \mathbb{G}$ The following statements are true:

1. $a * \mathcal{S}w$ - $Cl(B) \subseteq Cl(a * B)$ *and* $((\mathcal{S}w$ - $Cl(B)) * a \subseteq Cl(B * a))$.
2. $\mathcal{S}w$ - $Cl(a * B) \subseteq a * Cl(B)$ *and* $(\mathcal{S}w$ - $Cl(B * a) \subseteq (Cl(B) * a)$.
3. $A * \mathcal{S}w$ - $Cl(B) \subseteq Cl(A * B)$ *and* $((\mathcal{S}w$ - $Cl(B)) * A \subseteq Cl(B * A))$.

Proof.

1. Let y ∈ a ∗ $\mathcal{S}w$-$Cl(B)$ and let W be a $\mathcal{S}w$-open subset of $\mathbb{G}$, such that y ∈ W. Then, there is $x \in \mathcal{S}w$- $Cl(B)$ such that $y = a * x$. Since f is $\mathcal{S}w$-continuous, there exists a $\mathcal{S}w$-open set V in $\mathbb{G}$ such that $x \in V$ and $a * V \subseteq W$. Since $x \in V$ and $x \in \mathcal{S}w$-Cl(B) , then V ∩ B ≠ ∅, so there is, s ∈ V ∩ B Then, a ∗ s ∈ a ∗ V and a ∗ s ∈ a ∗ B, so (a ∗ V) ∩ (a ∗B) ≠ ∅. Hence, W ∩ (a ∗ B) ≠ ∅. This means that, ∈ $Cl(a * B)$. Thus, a ∗ $\mathcal{S}w$- $Cl(B) \subseteq Cl(a * B)$.

2. By (1), we have a^{-1} ($\mathcal{S}w$- ($Cla * B$)) ⊆Cl(a^{-1}∗($a * B$)) = Cl($a^{-1} * a) * B = Cl(B)$. Therefore, $a *$ ($a^{-1} *$ ($\mathcal{S}w$- $Cl(a * B)) \subseteq a * Cl(B)$. That is, $\mathcal{S}w$-$Cl(a * B) \subseteq a * Cl(B)$.

3. By (1) A ∗ $\mathcal{S}w$-$Cl(B) = \bigcup_{a \in A}(a * \mathcal{S}w\text{-}Cl(B)) \subseteq \bigcup_{a \in A} Cl(a * B) \subseteq Cl \bigcup_{a \in A}(a *$ B) = Cl(A ∗ B).

Theorem 2.25. Assume that ($\mathbb{G}$,∗.$\mathcal{T}$) is a $\mathcal{S}w$-topological group and the multiplication map f is a $\mathcal{S}w$-continuous. Then, for each $A.B \subseteq \mathbb{G}$ and $a.c \in \mathbb{G}$The following statements hold:

1. Int(a ∗ B) ⊆ a ∗$\mathcal{S}w$-Int(B) and (Int(B ∗ a) ⊆ ($\mathcal{S}w$-Int(B)) ∗ a).
2. a ∗ Int(B) ⊆ $\mathcal{S}w$-Int(a ∗ B) and ((Int(B)) ∗ a ⊆ $\mathcal{S}w$-Int(B ∗ a)).
3. A ∗ Int(B) ⊆ $\mathcal{S}w$-Int($A * B$) and ((Int(B)) ∗ A ⊆ $\mathcal{S}w$-Int($B * A$)).
4. B is a $\mathcal{S}w$- open if and only if $a * B$ is a $\mathcal{S}w$-open.
5. B is a $\mathcal{S}w$- closed if and only if $c * B$ is a $\mathcal{S}w$-closed.

Proof.

1. Let ∈ $Int(a * B)$. Then, there is an open set O in $\mathbb{G}$ such that $\in O \subseteq a * B$, then there is $b \in B$ such that $y = a * b$. By $\mathcal{S}w$-continuity of f, there exists a $\mathcal{S}w$-*open* subset V of G such that $b \in V$ and $a * V \subseteq O$, that is, $a * V \subseteq a * B$, so $a^{-1} * (a * V) \subseteq a^{-1} * (a * B)$, hence $V \subseteq B$. This means that, $b \in Int(B)$. $b \in \mathcal{S}w\text{-}Int(B)$, so $y = a * b \in a * \mathcal{S}w\text{-}Int(B)$. Hence $Int(a*B) \subseteq a* \mathcal{S}w\text{-}Int(B)$.

2. $a * Int(B) = a * Int(e * B) = a * (a^{-1} * (a * B)) \subseteq a * (a^{-1} \mathcal{S}w\text{-}Int(a * B)) = (a * a^{-1}) * \mathcal{S}w\text{-}Int(a * B) = e * \mathcal{S}w\text{-}Int(a * B) = \mathcal{S}w\text{-}Int(a * B)$.

3. $A* Int(B) = \bigcup_{a \in A} (a* Int(B)) \subseteq \bigcup_{a \in A} \mathcal{S}w\text{-}Int(a*B) \subseteq \mathcal{S}w\text{-}Int \bigcup_{a \in A} (a*B) = \mathcal{S}w\text{-}Int(A * B)$.

4. Let B be a $\mathcal{S}w$- open in. From Theorem 2.21, we have $l^{-1}{}_a$ is $\mathcal{S}w$- irresolute, so $(l_a{}^{-1})^{-1}$ (B) is a $\mathcal{S}w$ -open in $\mathbb{G}$. Since $(l_a{}^{-1})^{-1} = l_a$, so l_a (B) is a $\mathcal{S}w$-open in $\mathbb{G}$. Thus, $a * B$ is $\mathcal{S}w$- open in.

Conversely, let $a * B$ be a $\mathcal{S}$w-open in $\mathbb{G}$. From Theorem 2.21, we have l_a is a $\mathcal{S}w$-irresolute, then $l^{-1}{}_a(a * B)$ is a $\mathcal{S}w$-open in $\mathbb{G}$. Since $(l)^{-1}{}_a = l^{-1}{}_a$, so $l^{-1}{}_a(a * B)$ is a $\mathcal{S}w$-open in $\mathbb{G}$. Since $l^{-1}{}_a(a * B) = a^{-1} * (a * B) = B$, so B is a $\mathcal{S}w$-open in $\mathbb{G}$.

5. Let $\bar{B}$ be the complement of B So we get that $\bar{B}$ is a $\mathcal{S}w$-open set and by 4, $(a * \bar{B})$ is $\mathcal{S}w$-open in $\mathbb{G}$. thus $\overline{(a * \bar{B})} = (c * B) is\ a\ \mathcal{S}w$-open set in $\mathbb{G}$. Conversely, in the same way, we prove it.

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Proof. Without loss of generality, we assume that f is $\mathcal{S}w$-continuous. It is given that, $\mathcal{S}w\text{-} Int(S) = \emptyset$. Let $a.b \in \mathcal{S}w\text{-}Int(S)$ Then, there is a $\mathcal{S}w$-*open* subset V of $\mathbb{G}$ such that $b \in V \subseteq S$. Since S is a semigroup, so $a * b \in a * V \subseteq S$. But, from (4) of Theorem 2.25, we have *that*$*V$ is $\mathcal{S}w$-open in $\mathbb{G}$, so $a * b \in \mathcal{S}w\text{-}Int(S)$. Also, since $\mathcal{S}w\text{-}Int(S) \subseteq S$ and f is associative on S, so f is associative on $\mathcal{S}w\text{-}Int(S)$ Hence, $\mathcal{S}w\text{-}Int(S)$ It is a semigroup.

Theorem 2.27. Assume that $(\mathbb{G},*.\mathcal{T})$ is a $\mathcal{S}a$ *topological* group and the multiplication map f is $\mathcal{S}w$-continuous. If H is a semigroup subset of $\mathbb{G}$ and $\mathcal{S}w\text{-}Int(H) \neq \emptyset$. If is $\mathcal{S}w$- continuous, then $\mathcal{S}w\text{-}Int$(H) is a subgroup of $\mathbb{G}$.

Proof. We assume that f is $\mathcal{S}w$-continuous. By what we have done in the proof of Theorem 2.26, for *any* $a.b \in \mathcal{S}w\text{-}Int$(H) We obtain that a $*$ b $\in$ $\mathcal{S}w\text{-}Int$(H). Also, for any a $\in \mathcal{S}w\text{-}Int$(H), We have a $\mathcal{S}w$-open subset O of $\mathbb{G}$ such that a $\in$ O $\subseteq$ H, f: $\mathbb{G} \to \mathbb{G}$ is a bijective function, and it is $\mathcal{S}w$-continuous function. Since O is $\mathcal{S}w$-open in $\mathbb{G}$, so f^{-1} (V) $= \{ \; : f(x) \in V\} = \{x : x^{-1} \in V\} = O^{-1}$ *So* O^{-1} is $\mathcal{S}w$-open in $\mathbb{G}$. Since $a \in O \subseteq H$, so $a^{-1} \in O^{-1}a^{-1} \subseteq H^{-1} = H$. Hence $a^{-1} \in \mathcal{S}w\text{-}Int$(H) Therefore, $a * b^{-1} \in \mathcal{S}w\text{-}Int$(H), Hence $\mathcal{S}w\text{-}Int$(H) is a subgroup of $\mathbb{G}$.

Theorem 2.28. Assume that $(\mathbb{G},*.\mathcal{T})$ is a $\mathcal{S}a$ *topological* group and the multiplication map f is $\mathcal{S}w$-continuous. If H is a semigroup subset of $\mathbb{G}$ then $\mathcal{S}w\text{-} Cl(H)$ is a subgroup of $\mathbb{G}$.

Theorem 2.29. Assume that $(\mathbb{G},*.\mathcal{T})$ is a $\mathcal{S}a$ *topological* group and the multiplication map f is $\mathcal{S}w$-continuous. If (S) is a normal subset of $\mathbb{G}$ such that $\mathcal{S}w\text{-}Cl(S) \neq \emptyset$, then both $\mathcal{S}w\text{-}Cl(S)$ and $\mathcal{S}w\text{-}Int(S)$ are normal.

Proof. Let $x \in \mathbb{G}$, Then, $x^{-1} \in \mathbb{G}$. Since $\mathcal{S}w\text{-}Int(S)$ is $\mathcal{S}w$-open and f is a $\mathcal{S}w$-continuous at each point, then by (4) of Theorem 2.25 and as f is a $\mathcal{S}w$-continuous, we obtain that $x * \mathcal{S}w\text{-}Int(S) * x^{-1}$ is a $\mathcal{S}w$-open in $\mathbb{G}$ and $\mathcal{S}w\text{-}Int(x * \mathcal{S}w\text{-}Int(S) * x^{-1}) = x * \mathcal{S}w\text{-}Int(S) * x^{-1}$. Since S is a normal subgroup, so $x * \mathcal{S}w\text{-}Int(S) * x^{-1} \subseteq x * S * x^{-1} \subseteq S$, So $\mathcal{S}w\text{-}Int(x * \mathcal{S}w\text{-}Int(S) * x^{-1}) \subseteq \mathcal{S}w\text{-}Int(S)$. Therefore, $x * \mathcal{S}w\text{-}Int(S) \; x^{-1} \subseteq \mathcal{S}w\text{-}Int(S)$. Hence $\mathcal{S}w\text{-}Int(S)$ is a normal subset of $\mathbb{G}$ Now, we have to show that. $\mathcal{S}w\text{-}Cl(S)$ is also a normal subset of $\mathbb{G}$. To make this end, let $y \in x * \mathcal{S}w\text{-}Cl(S) * x^{-1}$ and O be any $\mathcal{S}w$-open subset of $\mathbb{G}$ such that y $\in O$. Then, there is s $\in \mathcal{S}w\text{-}Cl(S)$ such that y $= x * s * x^{-1}$ by Theorem 2.21. There exists a $\mathcal{S}w$-open subset V of $\mathbb{G}$ such that s $* x^{-1} \in$ V and $x * V \subseteq O$. Again, by Theorem 2.21, there is a $\mathcal{S}w$-open subset U in $\mathbb{G}$ such that s $\in U$ and $U * x^{-1} \subseteq$ V. That is, $x * U * x^{-1} \subseteq x * V \subseteq O$. Now, since $s \in U$ and s $\in \mathcal{S}w\text{-}Cl(S)$, then the U.S $\neq \emptyset$, So $(x * U * x^{-1}) \cap (x * S * x^{-1}) \neq \emptyset$. Since $(x * U * x^{-1}) \subseteq O$ and $x * S * x^{-1} \subseteq S$, So $O \cap S \neq \emptyset$. This implies that y $\in \mathcal{S}w\text{-}Cl(S)$ Thus, $x * \mathcal{S}w\text{-}Cl(S) * x^{-1} \subseteq \mathcal{S}w\text{-}Cl(S)$. Hence, $\mathcal{S}w\text{-}Cl(S)$ is a normal subset of $\mathbb{G}$.

CONCLUSION

In this paper, we introduce a quantization, a group and a reduction of continuity constraints that maintain crucial algebraic and topological interactions, thereby offering a useful generalization of topological groups. With important distinctions illustrated by counterexamples, the study emphasizes how these groups are very different from other generalized topological groups. It is demonstrated that not all. $\mathcal{S}$ w-topological groups have the same closure properties as traditional topological groups, but not all are semi-topological groups. Furthermore, homeomorphisms in these groups exhibit distinct behaviors. The publication raises several unanswered questions about the full extent of these groups and their potential uses, indicating that further research is needed to fully understand their significance in mathematical systems.

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