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On the Domination Numbers of Certain Prism Graphs

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ABSTRACT

A dominating set S of a graph $G = (V, E)$, is a subset of the vertex set *V (G)* such that any vertex not in S is adjacent to at least one vertex in S .The domination number of a graph G denoted by $\gamma\,(G)$ is the minimum size of the dominating sets of *G*. In this paper we introduced the domination numbers of certain prism graphs.

1. Introduction

Throughout this paper we consider simple graphs, finite, undirected and contain no loops or multiple edges. Our terminology and notations will be standard except as indicated. For undefined terms see [1], [2] and [3].

For a graph *G= (V, E)* , V denotes its vertex set while E its edge set. If $D \subseteq V$ then $< D >$ denotes the induced sub graph of G by the vertices of *D*. The cardinality of a set S denoted by $|S|$ is the number of elements of the *S*. A set $D \subseteq V$ is said to be dominating set of G if every vertex in $V - D$ is adjacent to some vertex in *D*. The cardinality of a minimum dominating set *D* is called the domination number of G and is denoted by $\gamma\,(G)$ [4]. In other words we defined the domination number $\gamma\,(G)$ of a graph G as the order of smallest dominating set of G. A dominating set S with $|S| = \gamma\,(G)$ is called a minimum dominating set [5]. Dominating set appear to have their origins in the game of chess where the goal is to cover or dominate various squares of a chessboard by certain chess pieces. The problem of determining domination numbers of graphs first emerged in (1862) of De Jaenisch who wanted to find the minimal number of queens on a chessboard, such that every square is either occupied by a queen or can be reached by a queen with single move [6]. To date many papers have been written on domination in graphs like [6,7,8]. In additional domination has many applications like the problem of monitoring an electric power system [9]. The book by Haynes, Hedetniemi, and Slater [2] illustrates many interesting examples including dominating queens, set of representatives, school but routing, computer communication networks, radio stations, land surveying …etc. Among them, the classical problems of covering chessboards by the minimum number of chess pieces were important in stimulating the study of domination [10].

As usual we use$\lceil x \rceil$ for the smallest integer not greater than *x*.

Definition 1.1: A Prism graph $Y_{m,n}$ is a simple graph given by the Cartesian product graph $Y_{m,n} = C_m \times P_n$ where C_m is a cycle with *m* vertices and P_n is a path with *n* vertices. It can therefore be viewed and formed by connecting n concentric cycle graphs C_m along spokes. Also sometime $Y_{m,n}$ calls a circular ladder graph [11].

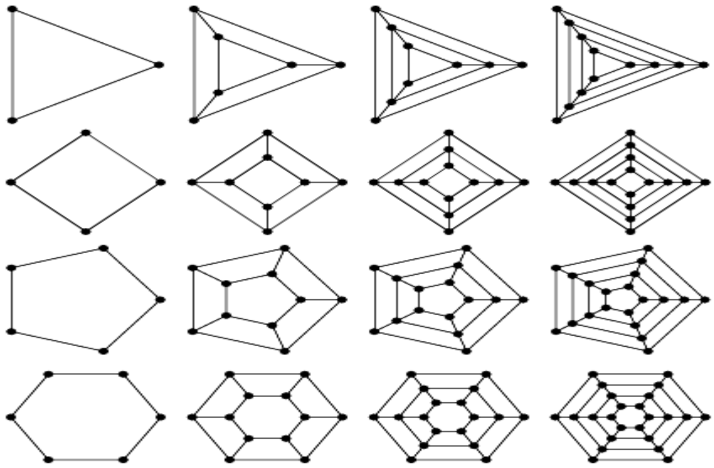

Fig. 1: $Y_{m,n}$

The following theorems have been used in this paper:

Theorem 1.1 (see [12]): If G has no isolated vertices, then $(G) \le \frac{|V(G)|}{2}$.

Theorem 1.2 (see [8]): A dominating set D of a graph G is minimal if and only if for each vertex $v \in D$ one of the following conditions satisfied

(i) There exist a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$

(ii) v is an isolated vertex in D .

2. On the domination numbers of prism graph $Y_{m,2}$

Theorem(2.1): For $m \ge 3$,

$$\gamma(Y_{m,2}) = \begin{cases} 2\left\lfloor\frac{m-2}{4}\right\rfloor + 3 & \text{for } m \equiv 1 \ (mod\ 4) \\ 2\left\lfloor\frac{m-2}{4}\right\rfloor + 2 & otherwise \end{cases}$$

Proof: Let the vertices of this graph labeled by: $V(Y_{m,2}) = \{1,2,3,\dots,2m\}$ see Fig.2.

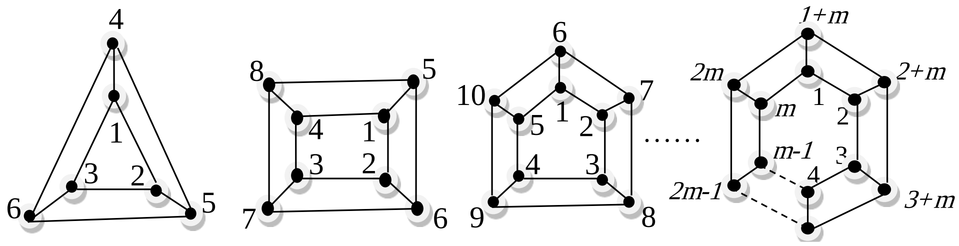

Fig. 2: $Y_{m,2}$

We consider for $m \ge 3$ the three following cases:

Case (1): Suppose $m \equiv 1\ (mod\ 4)$, we give the set of vertices $S = S_1 \cup S_2 \cup \{m\}$ is dominating set for $Y_{m,2}$.

Where $S_1 = \{1 + 4k \,|\, k = 0,1,2, \dots, \lfloor\frac{m-2}{4}\rfloor\}$

and $S_2 = \{(3 + m) + 4k \,|\, k = 0,1,2, \dots, \lfloor\frac{m-2}{4}\rfloor\}$.

$\therefore\ |S| = |S_1| + |S_2| + 1$

$= 2\lfloor\frac{m-2}{4}\rfloor + 3$

Therefore $\gamma(Y_{m,2}) \le |S| = 2\lfloor\frac{m-2}{4}\rfloor + 3$.

Now, to proof that $\gamma(Y_{m,2}) \ge 2\lfloor\frac{m-2}{4}\rfloor + 3$ it is sufficient to show that there is no proper subset of S dominating $Y_{m,2}$. So if v is any vertex in S, $S' = S - \{v\} \subset S$ be dominating $Y_{m,2}$ and $|S'| \le 2\lfloor\frac{m-2}{4}\rfloor + 2$. Now for any $v \in S$ implies that v either in S_1 or in S_2 or $v = \{m\}$. According to the structure of the labeling it is clear that for any chooses of $v \in S_1$ we have at least two vertices of the form $\{2 + 4k , m + 4k \,|\, k = 0,1,2, \dots, \lfloor\frac{m-2}{4}\rfloor\}$ that adjacent to v not dominating with any vertex in S'. For $v \in S_2$, v dominates itself and we have three vertices of the form: $\{3 + 4k , (m + 2) + 4k , (4 + m) + 4k \,|\, k = 0,1,2, \dots, \lfloor\frac{m-2}{4}\rfloor\}$ not dominating with any vertex in S' Also if $v = \{m\}$ then we have at least two vertices of the form $\{m - 1, 2m\}$ adjacent to v not dominating yet with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S', this implies that S' is not a dominating set or in other words we show that there is no proper subset $S' \subseteq S$ dominating $Y_{m,2}$ .Therefore we have a contradiction and $\gamma(Y_{m,2}) \ge 2\lfloor\frac{m-2}{4}\rfloor + 3$.

This implies that $\gamma(Y_{m,2}) = 2\lfloor\frac{m-2}{4}\rfloor + 3$.

Case(2): For $m \equiv 2\ (mod\ 4)$, we give the set of vertices $S = S_1 \cup S_2 \cup \{2m\}$ is the dominating set for $Y_{m,2}$ Where S_1 is the same set in case(1) and $S_2 = \{(3 + m) + 4k \,|\, k = 0,1,2, \dots, \lfloor\frac{m-2}{4}\rfloor - 1\}$

Therefore $|S| = |S_1| + |S_2| + 1$

$= 2\lfloor\frac{m-2}{4}\rfloor + 2$

Hence $\gamma(Y_{m,2}) \le |S| = 2\lfloor\frac{m-2}{4}\rfloor + 2$.

Similar to that in case(1), let $v \in S$, $S' = S - \{v\} \subset S$, $|S'| \le 2\lfloor\frac{m-2}{4}\rfloor + 1$ and S'be the dominating set of $Y_{m,2}$. Now if $v \in S_1$ or in S_2 the proof is similar to that in the case (1) remained when $v = \{2m\} \notin S'$, this vertex dominates itself which is not dominated with any vertex in $S' = S_1 \cup S_2$. So we have for all chooses of v at least one vertex not dominating with any vertex in S'.

$\therefore S'$ not dominating set of $Y_{m,2}$.

Hence $\gamma(Y_{m,2}) \geq |S| = 2\lfloor\frac{m-2}{4}\rfloor + 2$ implies that $\gamma(Y_{m,2}) = 2\lfloor\frac{m-2}{4}\rfloor + 2$.

Case (3): For $m \equiv 0,3\ (mod\ 4)$, we give the dominating set for these two cases of $Y_{m,2}$ by: $S = S_1 \cup S_2$ where S_1 and S_2 are the same sets in case(1).

$\therefore |S| = |S_1| + |S_2|$

$= 2\lfloor\frac{m-2}{4}\rfloor + 2$ Hence $\gamma(Y_{m,2}) \leq 2\lfloor\frac{m-2}{4}\rfloor + 2$.

The proof of $\gamma(Y_{m,2}) \geq 2\lfloor\frac{m-2}{4}\rfloor + 2$ is similar to that in case(1) when $v \in S_1$ or $v \in S_2$ $\therefore \gamma(Y_{m,2}) = 2\lfloor\frac{m-2}{4}\rfloor + 2$.

From the above cases we have:

$$\gamma(Y_{m,2}) = \begin{cases} 2\lfloor\frac{m-2}{4}\rfloor + 3 & \text{for } m \equiv 1\ (mod\ 4) \\ 2\lfloor\frac{m-2}{4}\rfloor + 2 & otherwise\,. \end{cases}$$ ■

3. On domination numbers of prism graph $Y_{m,3}$:

Theorem (2): For $m \geq 3$,

$$\gamma(Y_{m,3}) = \begin{cases} 3\lfloor\frac{m-2}{4}\rfloor + 4 & \text{for } m \equiv 1\ (mod\ 4) \\ 3\lfloor\frac{m-2}{4}\rfloor + 2 & \text{for } m \equiv 2\ (mod\ 4) \\ 3\lfloor\frac{m-2}{4}\rfloor + 3 & \text{for } m \equiv 0,3\ (mod\ 4) \end{cases}$$

Proof: Let the vertices of this graph labeled by: $V(Y_{m,3}) = \{1,2,3,\dots,3m\}$ see Fig.3.

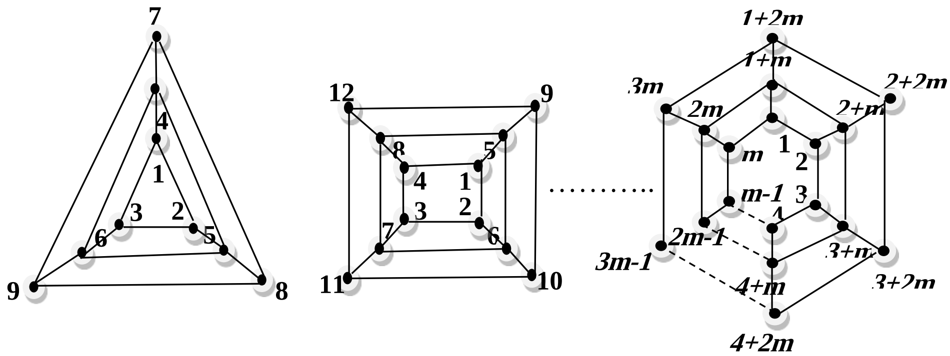

Fig. 3: $Y_{m,3}$

Similarly in Theorem(1) we consider for $m \geq 3$ the three following cases:

Case (1): Suppose $m \equiv 1\ (mod\ 4)$, first we give the set of dominating set of $Y_{m,3}$ by the set of the vertices $S = S_1 \cup S_2 \cup S_3 \cup \{2m-1\}$.

Where $S_1 = \{1 + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor\}$

$S_2 = \{(3+m) + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor\}$

and $S_3 = \{(1+2m) + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor\}$.

$|S| = |S_1| + |S_2| + |S_3| + 1$

$= 3\lfloor\frac{m-2}{4}\rfloor + 4$

Therefore $\gamma(Y_{m,3}) \leq |S| = 3\lfloor\frac{m-2}{4}\rfloor + 4$. By the same way as in Theorem (1) we prove $\gamma(Y_{m,3}) \geq 3\lfloor\frac{m-2}{4}\rfloor + 4$, let v any vertex in S , $S' = S - \{v\} \subset S$, $|S'| \leq 3\lfloor\frac{m-2}{4}\rfloor + 3$ and S' be the dominating set of $Y_{m,3}$. For any $v \in S$ implies that v is either in S_1 or in S_2 or in S_3 or $v = \{2m-1\}$. The proof is similar to that in Theorem(1), case(1) when $v \in S_1$ or S_2. So if $v \in S_3$ according to the structure of the labeling it is clear that v is dominates itself in addition there is at least one vertex of the form:

$\{2(m+1) + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor\}$ that adjacent to v not dominating with any vertex in S', remained when $v = \{2m-1\}$ we note v dominates itself and there are three vertices of the form $\{2m, m-1, 3m-1\}$that adjacent to this vertex not dominating with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S'.

$\therefore S'$ is not a dominating set implies $\gamma(Y_{m,3}) \geq 3\lfloor\frac{m-2}{4}\rfloor + 4$

Therefore $\gamma(Y_{m,3}) = 3\lfloor\frac{m-2}{4}\rfloor + 4$.

Case (2): For $m \equiv 2\ (mod\ 4)$.

We have give the set of vertices $S = S_1 \cup S_2 \cup S_3 \cup \{2m-1, 2(m-1)\}$

Where $S_1 = \{1 + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor - 1\}$

$S_2 = \{(3+m) + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor - 1\}$

and $S_3 = \{(1+2m) + 4k | k = 0,1,2,\dots,\lfloor\frac{m-2}{4}\rfloor - 1\}$

is a dominating set for $Y_{m,3}$.

Therefore $|S| = |S_1| + |S_2| + |S_3| + 2$

$= 3\lfloor\frac{m-2}{4}\rfloor + 2$

Hence $\gamma(Y_{m,3}) \leq |S| = 3\lfloor\frac{m-2}{4}\rfloor + 2$.

Now, let $v \in S$, $S' = S - \{v\} \subset S, |S'| \leq 3\lfloor\frac{m-2}{4}\rfloor + 1$and S'be the dominating set of $Y_{m,3}$. If $v \in S$ implies that v either in S_1or in S_2 or in S_3 or $v = \{2m-1, 2(m-1)\}$. The proof is similar to that in the case (1) for $v \in S_1$, $v \in S_2$and $v \in S_3$ so remained two vertices of the form $\{2m-1\}$ and $\{2m-2\}$. So either $v = \{2m-1\}$ that dominates itself in addition there are at least three vertices of the form $\{(m-1), (3m-1), 2m\}$ that adjacent to v not dominating with any vertex in S', or $v = \{2m-2\}$ that dominates itself and there are two vertices of the form $\{(m-2), (3m-2)\}$ not dominating with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S'

$\therefore S'$ is not a dominating set.

Hence $\gamma(Y_{m,3}) \geq 3\lfloor\frac{m-2}{4}\rfloor + 2$

This implies that $\gamma(Y_{m,3}) = 3\lfloor\frac{m-2}{4}\rfloor + 2$.

Case(3): For $m \equiv 0,3 \ (mod\ 4)$ give the set of dominating set of $Y_{m,3}$ by the set of the vertices $S = S_1 \cup S_2 \cup S_3$ where S_1, $S_2$and S_3 are the same sets in case(1).

Therefore $\gamma(Y_{m,3}) \leq |S| = 3\lfloor\frac{m-2}{4}\rfloor + 3$.

let $v \in S$, $S' \subset S$, $|S'| \leq 3\lfloor\frac{m-2}{4}\rfloor + 3$, the proof of $\gamma(Y_{m,3}) \geq 3\lfloor\frac{m-2}{4}\rfloor + 3$ is similar to that in case (1) when $v \in S_1$, $v \in S_2$ and $v \in S_3$ hence S' is not a dominating set, implies that $\gamma(Y_{m,3}) \geq 3\lfloor\frac{m-2}{4}\rfloor + 3$.

Hence from the above cases we have:

$$\gamma(Y_{m,3}) = \begin{cases} 3\lfloor\frac{m-2}{4}\rfloor + 4 & \text{for } m \equiv 1 \ (mod\ 4) \\ 3\lfloor\frac{m-2}{4}\rfloor + 2 & \text{for } m \equiv 2 \ (mod\ 4) \\ 3\lfloor\frac{m-2}{4}\rfloor + 3 & \text{for } m \equiv 0,3 \ (mod\ 4). \end{cases} \blacksquare$$

4. On domination numbers of prism graph $Y_{m,4}$:

In this section we first determine the domination number of $Y_{3,4}$, $Y_{5,4}$ and $Y_{9,4}$as aspecial cases then we determine the domination numbers of $Y_{m,4}$ in general.

Lemma(1): $\gamma(Y_{3,4}) = 4$

Proof: Let the vertices of this graph labeled as shown in Fig.4.

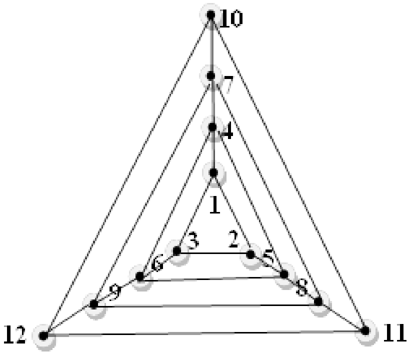

Fig. 4:$Y_{3,4}$

We have first give the dominating set of this graph by the

set $S = \{3,4,11,12\}$, therefore $\gamma(Y_{3,4}) \leq |S| = 4$

let $v \in S$, $S' \subset S$, $|S'| \leq 3$.

It is easy to show that there is no proper subset $S' = S - \{v\} \subseteq S$ dominating $Y_{3,4}$.

So if we choose $v \in S$ we have always at least one vertex that adjacent to $v$notdominating with any vertex in S'. So if $v = \{3\} \notin S'$, $S' = \{4,11,12\}$ it is clear $\{2,6\}$ not dominating with any vertex in S'. Now if $v = \{4\} \notin S'$, we have $\{7\}$ not dominating with any vertex in S'. If $v = \{11\} \notin S'$, we have $\{8\}$ not dominating with any vertex in S'. Also if $v = \{12\} \notin S'$, we have $\{9\}$ not dominating with any vertex in S'. So for any chooses of v we have always one vertex not dominating with any vertex in S'

$\therefore S'$ is not dominating set, hence we have a contradiction and $\gamma(Y_{3,4}) \geq 4$.

$\therefore \gamma(Y_{3,4}) = 4$. $\blacksquare$

Lemma(2): $\gamma(Y_{5,4}) = 6$

Proof: Let the vertices of this graph labeled by as shown in Fig.5.

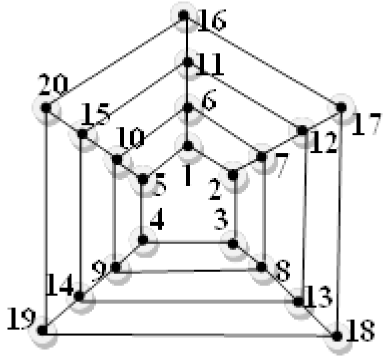

Fig. 5:$Y_{5,4}$

We have first give the dominating set of this graph by the set

$S = \{3,5,6,14,17,20\}$, therefore $\gamma(Y_{5,4}) \leq |S| = 6$

let $v \in S$, $S' = S - \{v\}$ and $S' \subset S$, therefore $|S'| \leq 5$

It is very easy as in lemma(1) to show that there is no proper subset $S' \subseteq S$ dominating $Y_{5,4}$. So if $v = \{3\}$, we have

three vertices $\{2,4,8\}$ not dominating with any vertex in S'. If $v = \{5\}$ not dominating with any vertex in S' because it is dominates itself. If $v = \{6\}$, we have two vertices $\{7,11\}$ not dominating with any vertex in S'.

If $v = \{14\}$, we have three vertices $\{9,13,15\}$ not dominating with any vertex in S'.

If $v = \{17\}$, we have three vertices $\{12,16,18\}$ not dominating with any vertex in S'.

If $v = \{20\}$, we have two vertices $\{15,16\}$ not dominating with any vertex in S'. So for any chooses of v we have always one vertices not dominating with any vertex in S'

$\therefore S'$ is not a dominating set. Hence $\gamma(Y_{5,4}) \geq 6$.

$\therefore \gamma(Y_{5,4}) = 6$. $\blacksquare$

Lemma(3): $\gamma(Y_{9,4}) = 10$

Proof: Let the vertices of this graph labeled as shown in Fig.6.

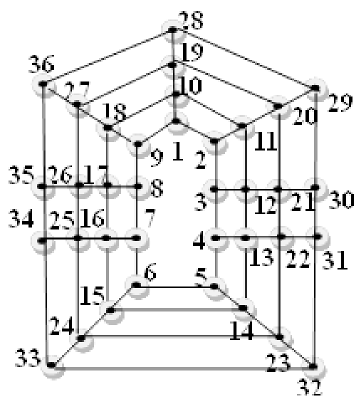

Fig. 6: $Y_{9,4}$

We have given the dominating set of this graph by the set S
where: $S = \{3,5,8,10,16,22,26,29,33,36\}$therefore $\gamma(Y_{9,4}) \le |S| = 10$.Similarly as in lemma(1) and lemma(2)
we have always v dominates itself or there is at least one
vertex that adjacent to v not dominating with any vertex in S'.
So no proper subset $S' \subseteq S$ dominating $Y_{9,4}$.
Hence $\gamma(Y_{9,4}) \ge 10$.
$\therefore \gamma(Y_{9,4}) = 10$. ■

Theorem(3): For $m \ge 4$ and $m \ne 5,9$, $\gamma(Y_{m,4}) = m$
Proof: Let the vertices of this graph labeled by:
$V(Y_{m,4}) = \{1,2,3,\dots,4m\}$ see Fig.7.

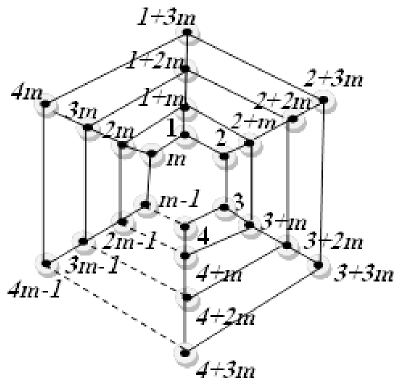

Fig. 7: $Y_{m,4}$

Similar to that in Theorem(1) and Theorem(2), we consider for $m \ge 4$ and $m \ne 5,9$ the four following cases:
Case(1): For $m \equiv 0 \ (mod\ 4)$
We give the set of dominating set of $Y_{m,4}$ by the set of the vertices:
$S = \{(i+m),(i+3m+1),(i+2),(i+2m+3) | i = 1,5,9,\dots,m-3\}$
$|S| = 4\left(\frac{m}{4}\right) = m$.
Therefore $\gamma(Y_{m,4}) \le |S| = m$
For any vertex v in S , $S' = S - \{v\} \subset S$, $|S'| \le m-1$ and S' be the dominating set of $Y_{m,4}$, according to the structure of the labeling if we take $v \in S$ we have four choices to v.
If $v = (i+m)$ the always three vertices are of the form $\{i, (i+m+1), (i+2m)\}$.
If $v = (i+3m+1)$ then, we have three vertices are of the form:
$\{(i+2m+1),(i+3m+2),(i+3m)\}$ not dominating with any vertex in S' .
If $v = (i+2)$ we have three vertices of the form $\{(i+1),(i+3),(i+m+2)\}$ not dominating with any vertex in S'.
If $v = (i+2m+3)$ we have two vertices are of the form:
$\{(i+3m+3),(i+2m+2)\}$ not dominating with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S'
Hence $\gamma(Y_{m,4}) \ge m$
$\therefore \gamma(Y_{m,4}) = m$.

Case (2): For $m \equiv 1 \ (mod\ 4)$ and note that $m \ne 5,9$ first we have give the set of dominating set of $Y_{m,4}$ by the set of the vertices $S = S_1 \cup S_2$
Where: $S_1 = \{(i+m),(i+3m+1),(i+2),(i+2m+3) | i = 8,12,16,\dots,m-5\}$
and $S_2 = \{3,5,(m+1),(2m+4),(3m+2),(m+7),(3m+6),(m-1),4m\}$.
$|S| = |S_1| + |S_2| = m$.
Therefore $\gamma(Y_{m,4}) \le |S| = m$.
Now, let v any vertex in S, $S' = S - \{v\} \subset S$, $|S'| \le m$ and S' be the dominating set of $Y_{m,4}$ .Similarly for any chooses $v \in S$ either in S_1 or in S_2.
For $v \in S_1$, v is dominates itself and all the vertices that adjacent to v except when $v = (8+m)$ we have three vertices of the form $\{8,(8+2m),(9+m)\}$ that adjacent to v not dominating with any vertex in S'.
Also, for $v \in S_2$, v is dominates itself and all the vertices that adjacent to v except when $v = \{(m-1),(m+7)\}$ we have two vertices of the form $\{m,(2m-1)\}$ and $\{(m+6),(2m+7)\}$ that adjacent to v not dominating with any vertex in S', similarly for any chooses of v we have at least two vertices not dominating with any vertex in S', so we have a contradiction, hence S' is not a dominating set and $\gamma(Y_{m,4}) \ge m$
$\therefore \gamma(Y_{m,4}) = m$.

Case (3): For $m \equiv 2 \ (mod\ 4)$ first we give the set of dominating set of $Y_{m,4}$ by the set of the vertices $S = S_1 \cup S_2$.
Where $S = \{(i+m),(i+3m+1),(i+2),(i+2m+3) | i = 1,5,9,\dots,m-5\}$
and $S_2 = \{(m-1),4m\}$.
$|S| = |S_1| + |S_2| = m$
Therefore $\gamma(Y_{m,4}) \le |S| = m$.
Now, let v any vertex in S , $S' = S - \{v\} \subset S$, $|S'| \le m-1$ and S' be the dominating set of $Y_{m,4}$.
For $v \in S_1$similarly, if $v = (i+m)$ or $v = (i+2)$ always v dominates itself and all the vertices that adjacent to him.
For $v = (i+3m+1)$ dominates itself and two vertices of the form:
$(i+2m+1),(i+3m+2)\}$ now, If $v = (i+2m+3)$ also dominates itself and two vertices of the form $\{(i+3m+3),(i+2m+2)\}$ not dominating with any vertex in S'. Now, for $v \in S_2$ implies that v either $(m-1)$or $4m$.
If $v = (m-1)$ we have two vertices of the form $\{(2m-1),m\}$ adjacent to v not dominating with any vertex in S'. Also if $v = \{4m\}$ we have two vertices of the form $\{(4m-1),3m\}$ adjacent to v not dominating with any vertex in S'. So for any chooses of v we have always two vertices not dominating with any vertex in S'.
Therefore S' is not a dominating set.
Hence $\gamma(Y_{m,4}) \ge m$.

This implies that $\gamma(Y_{m,4}) = m$.

Case(4): For $m \equiv 3\ (mod\ 4)$ first we give the set of dominating set of $Y_{m,4}$ by the set of the vertices $S = S_1 \cup S_2$.

Where $S = \{(i+m),(i+3m+1),(i+2),(i+2m+3) | i = 1,5,9,\dots,m-6\}$

and $S_2 = \{(m-2), 2m, (4m-1)\}$.

$$|S| = |S_1| + |S_2| = 4(\tfrac{m-3}{4}) + 3 = m$$

Therefore $\gamma(Y_{m,4}) \leq |S| = m$. Now, let v any vertex in S , $S' = S - \{v\} \subset S$, $|S'| \leq m-1$ and S' be the dominating set of $Y_{m,4}$ .If $v \in S$ implies that v either in S_1 or in S_2. for $v \in S_1$ according to the structure of the labeling for $v = (i+2)$ and

$v = (i+3m+1)$ it is clear that v is dominates itself and all the three vertices that adjacent to him.

If $v = (i+m)$ then, we have the three vertices are of the form:

$\{i, (i+m+1), (i+2m)\}$ not dominating with any vertex in S'.Now if

$v = (i+2m+3)$, v dominates itself and the vertex of the form $(i+3m+3)$.

Also if $v \in S_2$We have three choices:

If $v = (m-2)$ then, there are two vertices of the form $\{(2m-2),(m-1)\}$ that adjacent to v not dominating with any vertex in S'. For $v = (4m-1)$ then, we have all the vertices that adjacent to v not dominating with any vertex in S'. Now If $v = (2m)$ there are three vertices of the form $\{m, 3m, (2m-1)\}$ that adjacent to v not dominating with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S' so we have a contradiction.

Hence S' is not a dominating set implies $\gamma(Y_{m,4}) \geq m$

$\therefore \gamma(Y_{m,4}) = m$. ■

5. On domination numbers of prism graph $Y_{m,5}$:

As previous section we first determine the domination number of $Y_{3,5}$ and $Y_{6,5}$ as a special cases then we determined the domination numbers of $Y_{m,5}$ in general.

Lemma (4): $\gamma(Y_{3,5}) = 4$

Proof: Let the vertices of this graph labeled as shown in Fig.8.

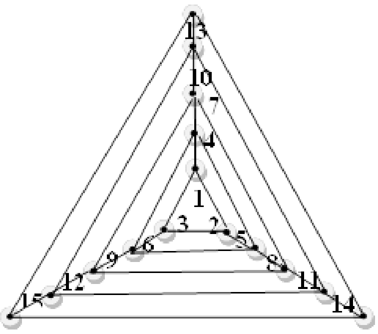

Fig. 8: $Y_{3,5}$

First we give the dominating set of this graph by the set S where: $S = \{3,7,8,15\}$, therefore $\gamma(Y_{3,5}) \leq |S| = 4$. For $v \in S$, $S' = S - \{v\} \subset S$, $|S'| \leq 3$.

It is easy to show that there is no proper subset $S' \subseteq S$ dominating

$Y_{3,5}$. So if $v = \{3\}$ then it dominates itself and there are three

vertices $\{1,2,6\}$ not dominating with any vertex in S'.

If $v = \{7\}$ then it dominates itself and the vertex $\{4\}$ that adjacent to v not dominating with any vertex in S'.

So if $v = \{8\}$ then it dominates itself and

there are $\{5,9,11\}$ not dominating with any vertex in S'.

Now, if $v = \{15\}$ then it dominates itself and there are $\{13,14\}$ not dominating with any vertex in S'. So for all chooses of v we have at least two vertices not dominating with any vertex in S'

$\therefore S'$ is not a dominating set.

We have a contradiction and $\gamma(Y_{3,5}) \geq 4$.

$\therefore \gamma(Y_{3,5}) = 4$. ■

Lemma (5): $\gamma(Y_{6,5}) = 8$

Proof: Let the vertices of this graph labeled as in Fig.9.

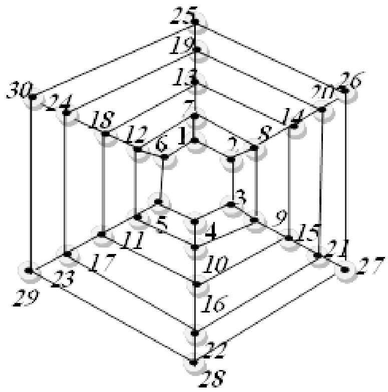

Fig. 9: $Y_{6,5}$

First we give the dominating set of this graph by:

$S = \{3,5,7,15,16,24,26,28\}$.

Therefore $\gamma(Y_{6,5}) \leq |S| = 8$.Similarly in lemma(4) for $v \in S$, $S' = S - \{v\} \subset S$, $|S'| \leq 7$,now, let v any

vertex in S implies that:

If $v = \{3\}$ in additional v dominates itself there is the vertex $\{2\}$

not dominating with any vertex in S'.

If $v = \{5\}$ then, there are two vertices $\{6,11\}$ that adjacent to v not dominating with any vertex in S'.

If $v = \{7\}$ then, there are four vertices $\{1,8,12,13\}$ that adjacent to v not dominating with any vertex in S'.

If $v = \{15\}$ then, there are two vertices $\{14,21\}$ that adjacent to v not dominating with any vertex in S'.

If $v = \{16\}$ then, there are two vertices $\{10,17\}$ that adjacent to v not dominating with any vertex in S'.

If $v = \{24\}$ then, there are four vertices $\{18,19,23,30\}$ that adjacent to v not dominating with any vertex in S'. If $v = \{26\}$ then, there are three vertices $\{20,25,27\}$ that adjacent to v not dominating with any vertex in S'. Now, if $v = \{28\}$ then, there is one vertex $\{29\}$ that adjacent to v not dominating with any vertex in S'. Hence S' is not dominating set since for all chooses of v we have at least one vertex not dominating with any vertex in S'.

So $\gamma(Y_{6,5}) \geq 8$ and $\gamma(Y_{6,5}) = 8$. ■

Theorem(4): For $m \geq 4$ and $m \neq 6$, $\gamma(Y_{m,5}) = \begin{cases} \frac{5m}{4} & \text{for } m \equiv 0 \ (mod\ 4) \\ \frac{5m+3}{4} & \text{for } m \equiv 1 \ (mod\ 4) \\ \frac{5m-2}{4} & \text{for } m \equiv 2 \ (mod\ 4) \\ \frac{5m+1}{4} & \text{for } m \equiv 3 \ (mod\ 4) \end{cases}$

Proof: Let the vertices of this graph labeled by: $V(Y_{m,5}) = \{1,2,3, \dots, 5m\}$ see Fig.10.

Similarly from above theorems we consider for $m \geq 4$ and $m \neq 6$ the four following cases:

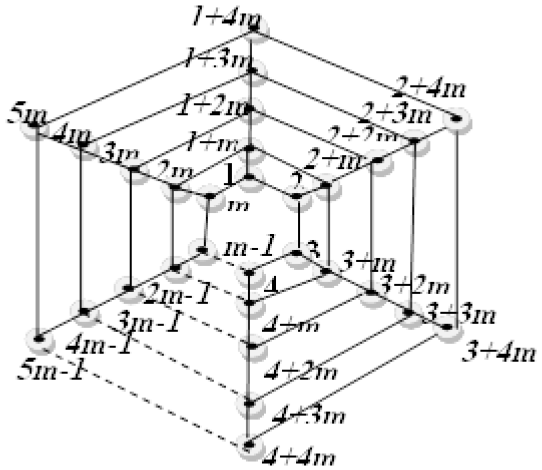

Fig. 10: $Y_{m,5}$

<u>Case (1):</u> For $m \equiv 0 \ (mod\ 4)$

First we give the set of dominating set of $Y_{m,5}$ by the set of the vertices S Where :

$S = \{(i+m), (i+4m+1), (i+2), (i+2m+1), (i+3m+3) | i = 1,5, \dots, m-3\}$

$\therefore |S| = \frac{5m}{4}$, therefore $\gamma(Y_{m,5}) \leq |S| = \frac{5m}{4}$

For any vertex v in S , $S' = S - \{v\} \subset S$, $|S'| \leq \frac{5m}{4} - 1$ and S' be the dominating set of $Y_{m,5}$. According to the structure of labeling if we take $v \in S$ then, we have five chooses:

For $v = \{(i+m), (i+2), (i+3m+3)\}$ we note v dominates itself and all the vertices that adjacent to v which is not dominated with any vertex in S'.

For $v = (i+2m+1)$ dominate itself and one vertex of the form $(i+2m+2)$ which is not dominating with any vertex in S'.Now, for $v = (i+4m+1)$ dominates itself and the two vertices of the form $\{(i+4m+2), (i+4m)\}$ that adjacent to v not dominating with any vertex in S'. So for any chooses of v we have at least two vertices not dominating with any vertex in S', hence S' is not dominating set implies $\gamma(Y_{m,5}) \geq \frac{5m}{4}$.

$\therefore \gamma(Y_{m,5}) = \frac{5m}{4}$.

<u>Case (2):</u> For $m \equiv 1 \ (mod\ 4)$

First we give the set of dominating set of $Y_{m,5}$ by the set of the vertices

$S = S_1 \cup \{2m, 5m\}$. Where

$S_1 = \{(i+2), (i+m), (i+2m+3), (i+3m+1), (i+4m+3) | i = 1,5, \dots, m-4\}$

$$|S| = |S_1| + 2 = 5\left(\frac{m-1}{4}\right) + 2 = \frac{5m+3}{4}$$

Therefore $\gamma(Y_{m,5}) \leq |S| = \frac{5m+3}{4}$. Now, let v be any vertex in S , $S' = S - \{v\} \subset S$, $|S'| \leq \frac{5m+3}{4} - 1$ and S' be the dominating set of $Y_{m,5}$ .Similarly in case(1) we have five chooses when $v \in S_1$.

For $v = \{(i+2), (i+3m+1)\}$ we note v dominates itself and all the vertices that adjacent to v which is not dominated with any vertex in S'.

For $v = (i+m)$we always at least one vertex of the form $(i+m+1)$ which is not dominating with any vertex in S'.

For $v = (i+2m+3)$ dominates itself and one vertex of the form $(i+2m+2)$ which is not dominated any vertex in S'.

Now, for $v = (i+4m+3)$ dominates itself and then, there are two vertices of the form $\{i+4(m+1), (i+4m+2)\}$ that adjacent to v not dominating with any vertex in S'.To complete the proof remained when $v = \{5m\}$ and $v = \{2m\}$ where in two chooses of v there is at least one vertex of the form $(m)$and $(4m)$ that adjacent to v not dominating with any vertex in S'. So for any chooses of v we have always one vertex not dominating with any vertex in S'.therefore S' is not a dominating set.

Hence $\gamma(Y_{m,5}) \geq \frac{5m+3}{4}$.

This implies that $\gamma(Y_{m,5}) = \frac{5m+3}{4}$.

<u>Case (3):</u> For $m \equiv 2 \ (mod\ 4)$and $m \neq 6$.

We give the set of dominating set of $Y_{,,5}$ by the set of the vertices $S = S_1 \cup S_2$

Where :

$S_1 = \{(i+2), (i+m), (i+2m+3), (i+3m+1), (i+4m+3) | i = 1,5, \dots, m-9\}$

and $S_2 = \{5m , (m-1), (3m-1), (2m-3) , (4m-4), (m-5), (5m-2)\}$.

$|S| = |S_1| + 7 = 5(\frac{m-6}{4}) + 7$

Therefore $\gamma(Y_{m,5}) \leq |S| = \frac{5m-2}{4}$

Now, let v any vertex in S , $S' = S - \{v\} \subset S$, $|S'| \leq \frac{5m-2}{4} - 1$ and S' be the dominating set of $Y_{m,5}$. The proof is similarly in the case (2) when $v \in S_1$ so to complete the proof remained when $v \in S_2$ hence if $v = \{5m\}$ which dominates itself and all the vertices that adjacent to v not dominating with any vertex in S'.

If $v = \{m-1\}$ which dominates itself then we have two vertices of the form

$\{m, m-2\}$ that adjacent to v not dominating with any vertex in S'.

If $v = \{3m-1\}$ which dominates itself then we have three vertices of the form $\{3m, (4m-$

$1),(3m-2)\}$ that adjacent to v not dominating with any vertex in S'.
If $v=\{m-5\}$ which dominates itself then we have two vertices of the form
$\{(m-4),(2m-5)\}$ that adjacent to v not dominating with any vertex in S'.
Finally for the other chooses of $v=\{(5m-2),(2m-3),(4m-4)\}$ which dominate itself and all the vertices that adjacent to v not dominating with any vertex in S'.Therefore S' is not a dominating set.
$\therefore S'$ is not a dominating set, so we have a contradiction and $\gamma(Y_{m,5})\geq\frac{5m-2}{4}$.
Implies that $\gamma(Y_{m,5})=\frac{5m-2}{4}$.

<u>Case (4):</u> For $m\equiv 3\ (mod\ 4)$.
We give the set of dominating set of $Y_{m,5}$ by the set of the vertices $S=S_1\cup S_2$ Where
$S_1=\{(i+2),(i+3m+1),(i+m),(i+2m+3),(i+4m+3)|i=1,5,\dots,m-6\}$
and $S_2=\{5m,(m-1),(3m-1),(3m-2)\}$.

$$|S|=|S_1|+|S_2|$$
$$=5\left(\frac{m-3}{4}\right)+4$$

Therefore $\gamma(Y_{m,5})\leq|S|=\frac{5m+1}{4}$.
Now, let v any vertex in S, $S'=S-\{v\}\subset S$, $|S'|\leq\frac{5m+1}{4}-1$ and S' be the dominating set of $Y_{m,5}$
If $v\in S$ implies that v either in S_1 or in S_2.
For $v\in S_1$ the proof is similarly in the case(1) so to complete the proof remained when $v\in S_2$ So we have four choices:
If $v=\{5m\}$ in additional that v dominates itself and all the vertices that adjacent to v not dominating with any vertex in S'.
If $v=\{m-1\}$ dominates itself then there are two vertices of the form $\{(m-2),m\}$ not dominating with any vertex in S'.
If $v=\{3m-1\}$ dominates itself then there are two vertices of the form
$\{(4m-1),3m\}$ not dominating with any vertex in S'.
If $v=\{3m-2\}$ dominates itself then there are two vertices of the form
$\{(4m-2),(2m-2)\}$ not dominating with any vertex in S'.
Hence S' is not dominating set. So $\gamma(Y_{m,5})\geq\frac{5m+1}{4}$.
$\therefore\gamma(Y_{m,5})=\frac{5m+1}{4}$. From the above cases we have:

$$\gamma(Y_{m,5})=\begin{cases}\frac{5m}{4} & \text{for } m\equiv 0\ (mod\ 4)\\ \frac{5m+3}{4} & \text{for } m\equiv 1\ (mod\ 4)\\ \frac{5m-2}{4} & \text{for } m\equiv 2\ (mod\ 4)\\ \frac{5m+1}{4} & \text{for } m\equiv 3\ (mod\ 4).\end{cases}\quad\blacksquare$$

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حول العدد المهيمن لبعض الحالات الخاصة للبيان الموشوري

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الملخص

تعرف المجموعة المهيمنة للبيان $G = (V, E)$ بأنها المجموعة الجزئية S من مجموعة الرؤوس V إذا كان لكل رأس ليس في S يجاور في الأقل رأسا واحداً في S. ويعرف العدد المهيمن Domination number الذي يرمز له $\gamma(G)$ بأنه حجم أصغر مجموعة ضمن المجموعات المهيمنة Dominating sets. في هذا البحث درسنا العدد المهيمن Domination number لبعض الحالات الخاصة للبيان الموشوري (Prism graph).